Topics to be covered on a PhD entrance exam in topology, Spring 2000

- Examples of topological spaces.
- Separation axioms (T_0-, T_1-, Hausdorff, regular, and normal spaces).
- Metric space topology (completeness, equivalent forms of compactness).
- Continuity.
- Connected spaces.
- Compactness.

Suggested reference books.

- Dugundji, *Topology*, Allyn & Bacon. (Chapters I-IX and XI.)
- Kelly, *General Topology*, D. van Nostrand. (Chapters: all except II, VI, and Appendix.)
- Gemignani, *Elementary Topology*, Addison-Wesley. (Chapters: all except XI.)

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Topology Ph.D. Entrance Exam, August 2000

In the exercises that follow $\overline{A}$ stands for the closure of A, and $A \setminus B$ for the set difference: $A \setminus B = \{x \in A \colon x \notin B\}$.

Ex. 1. (a) Define a T_0 topological space.

(b) Show that a topological space X is a T_0-space if and only if $\overline{\{x\}} \neq \overline{\{y\}}$ for every distinct $x, y \in X$.

Ex. 2. A topological space X is said to be *completely regular* provided that for each $p \in X$ and closed set A in X such that $p \notin A$, there is a continuous function $f \colon X \to [0,1]$ such that $f(p) = 0$ and $f[A] = \{1\}$.

Prove that any subspace of a completely regular space is completely regular.

Ex. 3. Let X be a topological space and let A and B be non-empty proper closed subsets of X such that $X = A \cup B$. Show that $X \setminus (A \cap B)$ is not connected.

Ex. 4. (a) Give an example of sets A_i $(i = 1, 2, 3, \ldots)$ in a topological space for which

$$\overline{\bigcup_{i=1}^{\infty} A_i} \neq \bigcup_{i=1}^{\infty} \overline{A_i}.$$

(b) Show that for any family $\{A_i \colon i = 1, 2, 3, \ldots\}$ of subsets of a topological space X the following formula holds:

$$\overline{\bigcup_{i=1}^{\infty} A_i} = \bigcup_{i=1}^{\infty} \overline{A_i} \cup \bigcap_{k=1}^{\infty} \overline{\bigcup_{i=k}^{\infty} A_i}.$$

Ex. 5. Let $S = \langle \mathbb{R}, \tau_S \rangle$ be a Sorgenfrey line, $D(\mathbb{N}) = \langle \mathbb{N}, \tau_D \rangle$ be a discrete topology on $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $D(\mathbb{R}) = \langle \mathbb{R}, \tau_D \rangle$ be a discrete topology on $\mathbb{R}$.

Show that there is a continuous mapping from S *onto* $D(\mathbb{N})$ but that there is no continuous mapping from S *onto* $D(\mathbb{R})$.

Ex. 6. Let X be a normal space and let U_1 and U_2 be open subsets of X such that $X = U_1 \cup U_2$. Show that there are open sets V_1 and V_2 such that $\overline{V_1} \subset U_1$, $\overline{V_2} \subset U_2$, and $X = V_1 \cup V_2$.

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Topology Ph.D. Entrance Exam, April 2001

Ex. 1. Let $\langle X_0, \tau_0 \rangle$ and $\langle X_1, \tau_1 \rangle$ be connected topological spaces. Show that $X_0 \times X_1$ with the product topology is connected.

Ex. 2. Consider the real line $\mathbb{R}$ with the topology τ generated by the family of intervals:

$$\mathcal{F} = \{[a, b) \colon a \in \mathbb{Q} \ \& \ b \in \mathbb{R} \ \& \ a < b\},$$

where $\mathbb{Q}$ stands for the set of rational numbers. Let X be the product of $\langle \mathbb{R}, \tau \rangle$ with itself (with the product topology). Prove or disprove that X is normal.

Ex. 3. Prove or find a counterexample for the statement:

A compact subset of a topological space $\langle X, \tau \rangle$ is closed in X.

Ex. 4. Let τ be the usual topology on the real line $\mathbb{R}$. Answer one of the following two questions.

(a) Does there exists a topology $\tau_0 \subset \tau$ such that $\langle \mathbb{R}, \tau_0 \rangle$ is homeomorphic to figure eight (i.e., two circles tangent at a point)?

(b) Does there exists a topology $\tau_0 \subset \tau$ such that $\langle \mathbb{R}, \tau_0 \rangle$ is homeomorphic to the unit circle $S^1 = \{\langle x, y \rangle \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$?

Ex. 5. Let $\langle X, \tau \rangle$ and $\langle Y, \tau' \rangle$ be the topological spaces and let $f \colon X \to Y$ be a function. Consider the graph $G(f) = \{\langle x, f(x) \rangle \colon x \in X\}$ of f as a subspace of the cartesian product $X \times Y$ (with the product topology). Prove or disprove each the following.

(a) If f is continuous, then $G(f)$ is homeomorphic to X.

(b) If $G(f)$ is homeomorphic to X, then f is continuous.

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Topology Ph.D. Entrance Exam, August 2001

Ex. 1. Let $\mathbb{R}^2$ be the euclidean plane (i.e., with natural topology). Let

$$X = \{\langle x, y\rangle \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{\langle x, 0\rangle \in \mathbb{R}^2 : -1 \le x \le 1\},$$

$$Y = \{\langle x, y\rangle \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{\langle x, 0\rangle \in \mathbb{R}^2 : -1 \le x \le 2\}.$$

Are X and Y homeomorphic? Give reasons for your answer.

Ex. 2. Prove that every compact metric space has a countable base for its topology.

Ex. 3. Let $\langle X, d\rangle$ be a compact metric space, and let $f\colon X \to X$ satisfy

$$d(f(x_1), f(x_2)) < d(x_1, x_2) \quad \text{for all distinct } x_1, x_2 \in X.$$

Show that there is a point $p \in X$ such that $f(p) = p$.

Ex. 4. A topological space X is said to have *countable pseudo character* provided every singleton in X is a G_δ-set (i.e., it is a countable intersection of open sets). Show that every compact Hausdorff space with countable pseudo character is first countable, that is, it has a countable local base at every point $x \in X$.

Ex. 5. Let $\mathcal{F}$ be the family of all *non-zero* polynomials of the form

$$w(x, y) = a_0x^2 + a_1y^2 + a_2xy + a_3x + a_4y + a_5$$

with rational coefficients and for every $w \in \mathcal{F}$. Let

$$E_w = \{\langle x, y\rangle \in \mathbb{R}^2 : w(x, y) = 0\}.$$

Show that the plane $\mathbb{R}^2$ is not covered by the sets E_w with $w \in \mathcal{F}$, that is, that $\mathbb{R}^2 \ne \bigcup_{w\in\mathcal{F}} E_w$.

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Topology Ph.D. Entrance Exam
Fall 2004

(1) Let $(\mathbb{R}, T)$ be the Sorgenfrey line (i.e. the set of the real numbers, $\mathbb{R}$, with the topology generated by $\{[a,b) | a, b \in \mathbb{R} \text{ with } a < b\}$).
   (a) Prove or disprove: $(\mathbb{R}, T)$ is homeomorphic to $\mathbb{R}$ in the standard topology generated by $\{(a,b) | a, b \in \mathbb{R} \text{ with } a < b\}$.

   (b) Show that the Sorgenfrey line is first countable, but not metrizable.

(2) Let (X, T) be a topological space, and let (Y, T') be a Hausdorff space. For a function $f : X \to Y$, let $G(f)$ denote the graph of f; specifically,

$$G(f) = \{(x, f(x)) \in X \times Y : x \in X\}$$

where $X \times Y$ is the Cartesian product with the product topology.
   (a) Prove that if $f : X \to Y$ is continuous, then $G(f)$ is a closed subset of $X \times Y$.

   (b) Give an example of a function f from the reals $\mathbb{R}^1$ (standard topology) to the reals $\mathbb{R}^1$ (standard topology) such that the graph $G(f)$ is closed in the plane $\mathbb{R}^1 \times \mathbb{R}^1$ but such that f is not continuous.

(3) Show that $\mathbb{R}$ and $\mathbb{R}^n$ (in the usual topologies) are not homeomorphic if $n > 1$.

(4) The Cartesian product of any two connected topological spaces is connected.

(5) Answer T for true, F for false. Indicate reasons for your answers without details.
   (a) ☐ The Cartesian product of any two normal Hausdorff spaces is normal.

   (b) ☐ Every topological space is a continuous image of a metric space.

   (c) ☐ Every Hausdorff space is regular.

   (d) ☐ The intersection of two connected metric spaces is connected.

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Topology Ph.D. Entrance Exam
Spring 2005

Choose four (and only four) of the following problems and work them neatly on each page on which the problem is stated. You will be provided with scratch paper for calculation and preliminary work, but this should not be turned in. All work to be graded should be written up in readable form on the paper provided.

(1) Prove that if (X, T_1) and (Y, T_2) are connected topological spaces, then $X \times Y$ with the product topology T is connected.

(2) Let (X, T) be a Hausdorff space, and let $p_1, p_2, \ldots, p_n$ be finitely many distinct points of X. Prove that there are open subsets $U_1, U_2, \ldots, U_n$ of X such that $p_i \in U_i$ for all i and $U_i \cap U_j = \phi$ for all $i \neq j$.

(3) Let S be the Sorgenfrey topology for the real line $\mathbb{R}$ (i.e., S is generated by all intervals for the form $[a, b)$ for $a, b \in \mathbb{R}$). Let $\mathbb{Q}$ denote the rational numbers with their usual topology T (i.e. $\mathbb{Q}$ has the subspace topology determined by the usual topology on $\mathbb{R}$).

(a) Is there a continuous function from $(\mathbb{R}, S)$ onto $(\mathbb{Q}, T)$? Justify your answer.

(b) Is there a continuous function from $(\mathbb{R}, S)$ onto $\mathbb{R}$ with the usual topology? Justify your answer.

(4) Let (X, d) be a compact metric space, and let $\mathcal{U}$ be an open cover of X. Show that there exists $\epsilon > 0$ such that for each $x \in X$, there exists $U_x \in \mathcal{U}$ such that the open ball $B_d(x, \epsilon) \subset U_x$.

(5) Let (X, d_1) and (Y, d_2) be metric spaces with their usual topologies T_{d_1} and T_{d_2}, respectively. Let $p \in X$ and let $f : X \to Y$ be a function. Prove that f is continuous at p if and only if f is sequentially continuous at p (i.e., for each sequence $\{s_n\}_{n=1}^{\infty}$ converging to p, the sequence $\{f(s_n)\}_{n=1}^{\infty}$ converges to $f(p)$).

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Topology Ph.D. Entrance Exam
Fall 2005

Work each individual problem including subparts and choose one from each of the following grouped problems neatly on each page on which the problem is stated. Work only three problems total. You will be provided with scratch paper for calculation and preliminary work, but this should not be turned in. All work to be graded should be written up in readable form on the paper provided.

Group 1 (choose one)

(1) Let (X, T) be a topological space, and let $A \subset X$. Let χ_A denote the characteristic function for A (i.e., $\chi_A(x) = 0$ if $x \in X - A$ and $\chi_A(x) = 1$ if $x \in A$). What conditions on A are both necessary and sufficient for χ_A to be continuous ($\{0, 1\}$ has the usual topology)? Prove your answer is correct.

(2) Let X be a metric space with its usual topology T.
   (a) Prove that if (X, T) is separable, then every collection of mutually disjoint open subsets of X is countable.
   (b) Is the converse of (a) true?

Group 2 (choose one)

(3) Let $Y \subset X$ and let X and Y be connected. Show that if A and B form a separation of $X - Y$, then $Y \cup A$ and $Y \cup B$ are connected.

(4) Show that if X is normal, then the members of any pair of disjoint closed sets have neighborhoods whose closures are disjoint.

(5) Let (X, T) be a topological space, let Y be a set, and let f be a function from X onto Y. Let

$$T_f = \{U \subset Y : f^{-1}(U) \in T\}.$$

(a) Prove that T_f is a topology.

(b) Let (Z, T_Z) be a topological space, and let $g : Y \to X$ be a function. Prove that if the composition $g \circ f$ is $T - T_Z$ continuous, then g is $T_f - T_Z$ continuous.

(c) Let X be the interval $[0, 2\pi]$ with the usual topology T, and let Y be the set whose members are the following subsets of X : $\{x\}$ if $0 < x < 2\pi$ and $\{0, 2\pi\}$. Let $f : X \to Y$ be the natural map (i.e., $f(x)$ is the member of Y containing x). Let T_f be as in (a). What familiar space do you think (Y, T_f) is? Verify your answer (hint: make use of (b)).