Problems with lines and planes:

Plane thru three points: If the points are P,Q,R then $\left(\vec{Q}-\vec{P}\right)\times\left(\vec{R}-\vec{P}\right)=\vec{n}$ gives the normal to the plane. Then use any one of the three given points to write $(\vec{r}-\vec{r}_0)\bullet\vec{n}=0$ as the equation of the plane.

Example: $P=(-1,2,2),\ Q=(3,-2,5),\ R=(-2,1,3)$

$\vec{Q}-\vec{P}=<4,-4,3>$, $\vec{R}-\vec{P}=<-1,-1,1>$

$\left(\vec{Q}-\vec{P}\right)\times\left(\vec{R}-\vec{P}\right)=<4,-4,3>\times<-1,-1,1>=\langle -1,-7,-8\rangle=\vec{n}$

We'll use $\vec{n}=<1,7,8>$ for convenience; we just need any vector normal to the plane. Using P : $(\vec{r}-\vec{r}_0)\bullet\vec{n}=1(x+1)+7(y-2)+8(z-2)=0,$ or $x+7y+8z=29$ is then the equation of the plane. Check it, it works!

Intersection of line with plane: If the line is in parametric form $\vec{r}=\vec{r}_0+t\vec{v}$ (or in scalar parametric form) and the plane is in the form of an equation $\left(\vec{r}-\vec{P}_0\right)\bullet\vec{n}=0$ or $ax+by+cz=d$ then plug the parametric forms for the line into the equation of the plane, solve for $t,$ and then find the corresponding values of $x,y,z.$

Example: Find the intersection of $\vec{r}=<3,-1,2>+t<5,2,3>$ with the plane $3x-y+2z=3.$ From the equation of the line, we have $x=3+5t, y=-1+2t, z=2+3t$ and plugging in to the equation of the plane, we have t satisfying $3(3+5t)-(-1+2t)+2(2+3t)=3$ when the line hits the plane. We solve for $t,$ obtaining $t=-\frac{11}{19}$, and so

$x=3+5\left(-\frac{11}{19}\right)=\frac{2}{19}$, $y=-1+2\left(-\frac{11}{19}\right)=-\frac{41}{19}$, $z=2+3\left(-\frac{11}{19}\right)=\frac{5}{19}$

is the point of intersection. Check it, it works! Note that we can also proceed in vector form: The plane is $\vec{r}\bullet<3,-1,2>=3$ and plugging in $\vec{r}=<3,-1,2>+t<5,2,3>$ we obtain $(<3,-1,2>+t<5,2,3>)\bullet<3,-1,2>=3$ and we can solve for t as before.

Intersection of two planes: Solve the two equations simultaneously and obtain two different solutions, corresponding to two points on the line of intersection. Then write the equation of the line. Or if you find the general solution by Gaussian elimination, you will immediately obtain the equation of the line.

Ex: $x-2y+3z=1$, $2x+y+2z=3$ are the two planes.

The augmented matrix is

$$\left[A\,\vdots\,\vec{b}\right]=\begin{bmatrix}1 & -2 & 3 & 1\\ 2 & 1 & 2 & 3\end{bmatrix}\sim\begin{bmatrix}1 & 0 & \frac{7}{5} & \frac{7}{5}\\ 0 & 1 & -\frac{4}{5} & \frac{1}{5}\end{bmatrix}$$ and the solution is

$x=\frac{7}{5}-\frac{7}{5}t$, $y=\frac{1}{5}+\frac{4}{5}t$, $z=t$ which can also be written in vector form.

Distance of a point to plane: The formula we will obtain is $D=\frac{|ax+cy+cz-d|}{\sqrt{a^2+b^2+c^2}}$ where the equation of the plane is $ax+by+cz=d.$

The meaning of this distance is that we drop a perpendicular (called the normal line) to the plane from the point and then find the distance from the given point to the point of intersection. There are two approaches: One is the direct approach where we find where the normal line intersects the plane; the other is a nice formula which bypasses the need to find the point of intersection.

Given point P and plane $(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$

Method 1: Normal line: $\vec{r} = \vec{P} + t\vec{n}$. Plug into equation of plane: $\left(\vec{P} + t\vec{n} - \vec{r}_0\right) \cdot \vec{n} = 0$, $t(\vec{n} \cdot \vec{n}) = \left(\vec{r}_0 - \vec{P}\right) \cdot \vec{n}$, $t = \dfrac{\left(\vec{r}_0 - \vec{P}\right) \cdot \vec{n}}{\vec{n} \cdot \vec{n}}$ and so the point of intersection (actually the position vector of the point of intersection), which is also the point on the plane closest to the given point, is given by $\vec{r} = \vec{P} + t\vec{n} = \vec{P} + \dfrac{\left(\vec{r}_0 - \vec{P}\right) \cdot \vec{n}}{\vec{n} \cdot \vec{n}}\vec{n}$. The distance D from $\vec{P}$ is $\left\| \dfrac{\left(\vec{r}_0 - \vec{P}\right) \cdot \vec{n}}{\vec{n} \cdot \vec{n}}\vec{n} \right\| = \dfrac{\left|\left(\vec{r}_0 - \vec{P}\right) \cdot \vec{n}\right|}{\|\vec{n}\|} = D.$

In a particular problem, one would carry out the calculations for the data at hand (see example), but it is instructive how these calculations can be done using vector operations throughout. The last formula is very simple and can be directly obtained as shown in method 2.

Method 2: Consider the vector $\vec{r}_0 - \vec{P}$ from the given point to a point on the plane. The component of this vector in the direction of the normal gives, through its absolute value, the distance of $\vec{P}$ from the plane. This is given by $D = comp_{\vec{n}}\left(\vec{r}_0 - \vec{P}\right) = \dfrac{\left|\left(\vec{r}_0 - \vec{P}\right) \cdot \vec{n}\right|}{|\vec{n}|}$. If the plane has the form $\vec{r} \cdot \vec{n} = d$ then this formula becomes $D = \dfrac{\left|d - \vec{P} \cdot \vec{n}\right|}{|\vec{n}|}$. Finally, if we put $P = (x,y,z)$ and $\vec{n} =< a,b,c >$ we obtain the form $D = \dfrac{|ax + cy + cz - d|}{\sqrt{a^2 + b^2 + c^2}}$. Note that the equation of the plane is $ax + cy + cz - d = 0$ so if (x,y,z) happens to be on the plane, then we get $D = 0$ as we should.

Finding a point on a plane closest to a given point P: This is discussed above, where we find the intersection of the normal line from P with the plane.

Ex: a) Find the point on the plane $2x + y - 3z = 4$ closest to $P = (-1,2,1)$

The normal line from P is $x = -1 + 2t$, $y = 2 + t$, $z = 1 - 3t$ and plugging in $2(-1 + 2t) + (2 + t) - 3(1 - 3t) = 4$, $-3 + 14t = 4$, $14t = 7$, $t = 1/2$ and $(x,y,z) = (0,5/2,-1/2)$ is the closest point on the plane to P.

b) Find the distance from the plane $2x + y - 3z = 4$ to $P = (-1,2,1)$.

$D = \dfrac{|2(-1) + 2 - 3 - 4|}{\sqrt{4 + 1 + 9}} = \frac{1}{2}\sqrt{14}$ gives us the distance directly. Note that, using the answer from part a), we could also calculate $D = |< -1,2,1 > - < 0,5/2,-1/2 >| = \frac{1}{2}\sqrt{14}$ but the calculation of part a) is not necessary.

Distance from origin to plane: $D = \frac{|d|}{|\vec{n}|}$, a special case of the previous formula with $P = (0,0,0)$

Dist between 2 nonparallel lines: If the lines are $L_1 : \vec{r} = \vec{P}_1 + \vec{v}_1 t$ and $L_2 : \vec{r} = \vec{P}_2 + \vec{v}_2 t$ then the formula is $D = \frac{\left|\left(\vec{P}_2 - \vec{P}_1\right) \cdot \vec{n}\right|}{|\vec{n}|}$ where $\vec{n} = \vec{v}_1 \times \vec{v}_2$.

The lines can be embedded in parallel planes with common normal $\vec{n} = \vec{v}_1 \times \vec{v}_2$. The planes are $\left(\vec{r} - \vec{P}_1\right) \cdot \vec{n} = 0$ and $\left(\vec{r} - \vec{P}_2\right) \cdot \vec{n} = 0.$ The distance between these planes is the distance between the lines, and the formula above then follows.

If the lines are parallel, use the distance from a point to a line formula below.

Closest points on 2 lines:

We want to find $\vec{P} = \vec{P}_1 + \vec{v}_1 t$ on L_1 and $\vec{Q} = \vec{P}_2 + \vec{v}_2 s$ on L_2 such that $\vec{Q} - \vec{P}$ is perpendicular to both lines. So we obtain the two equations:
$\left[\left(\vec{P}_2 + \vec{v}_2 s\right) - \left(\vec{P}_1 + \vec{v}_1 t\right)\right] \cdot \vec{v}_1 = 0$, $\left[\left(\vec{P}_2 + \vec{v}_2 s\right) - \left(\vec{P}_1 + \vec{v}_1 t\right)\right] \cdot \vec{v}_2 = 0$, which, when we solve for s and $t,$ will give us the desired points.

Distance of a point to a line: Given point P and line $\vec{r} = \vec{P}_0 + \vec{v}t$, the formula here is $D = \frac{\left|\left(\vec{P} - \vec{P}_0\right) \times \vec{v}\right|}{|\vec{v}|}.$ This formula applies because $D = \left|\vec{P} - \vec{P}_0\right| \sin\theta$, where θ is the angle between $\vec{P} - \vec{P}_0$ and the line.

Closest pt on line to given point: Given point P and a line $\vec{r} = \vec{P}_0 + t\vec{v}$ we arrive at the point on the line closest to P when $\vec{P} - \vec{r}$ is perpendicular to $\vec{v}.$ So we want $\left(\vec{P} - \vec{r}\right) \cdot \vec{v} = 0$ or $\left(\vec{P} - \vec{P}_0 - t\vec{v}\right) \cdot \vec{v} = 0$. This equation we solve for t and so obtain the desired point on the line.