Linear independence:

We say that a set (usually a finite set) of vectors is linear independent if the only linear combination of the vectors that produces the zero vector is when all the coefficients are zero. Mathematically, we say:

> The set of vectors $\bar{v}_1$, $\bar{v}_2,..,\bar{v}_k$ is linearly independent if $c_1\bar{v}_1 + c_2\ \bar{v}_2 +..+c_k\bar{v}_k = \bar{0}$ only when $c_1 = 0$, $c_2 = 0$,$\ldots$,$c_k = 0$.

We often say colloquially "the vectors $\bar{v}_1$, $\bar{v}_2,..,\bar{v}_k$ are linearly independent" rather than employing the formally correct "set of vectors" formulation. We'll also often simply say "independent" to mean "linearly independent".

If a set of vectors is not linearly independent, we say (not surprisingly) that the set is linearly dependent. If a set is linearly dependent, then, essentially, one vector can be expressed in terms of the others via a linear combination. In this case, since $c_1\bar{v}_1 + c_2\ \bar{v}_2 +..+c_k\bar{v}_k = \bar{0}$ with not all the $c's$ being zero, you can divide by a nonzero coefficient and isolate its corresponding vector on one side, expressed in terms of the other vectors on the other side. For instance, if $2\bar{v}_1 + 3\bar{v}_2 + 0\bar{v}_3 + 0\bar{v}_4 + 4\bar{v}_5 = \bar{0}$, we have, $\bar{v}_2 = -\frac{2}{3}\bar{v}_1 - \frac{4}{3}\bar{v}_5$. Of course we could do the same kind of thing other ways, for instance expressing $\bar{v}_1$ in terms of $\bar{v}_2$ and $\bar{v}_5$. That's one reason that the definition for linear indepence above is the one used, in that all vectors are treated equally. Another reason for the defintion is to take care of the case when there is a single vector in the set: $\{\bar{v}\}$ is linearly independent as long as $\bar{v} \neq \bar{0}$, but if $\bar{v} = \bar{0}$ the set is linearly dependent since $1\bar{v} = \bar{0}$. But you couldn't really say that in the set $\{\bar{0}\}$ the vector $\bar{0}$ can be expressed in terms of the other vectors in the set, because there aren't any other vectors! Note that any set of vectors containing the zero vector is linearly dependent for the same reason, since $1\bar{0} = \bar{0}$ represents a linear combination of the vectors in the set giving the zero vector without the coefficients being all zero.

The idea of linear independence is that if we think of all the vectors that can be generated by all possible linear combinations of a set of vectors, when the set is linearly dependent we can "get rid of" one of the vectors in the set and still generate the same vectors through linear combinations. But when the set is linearly independent, none of the vectors are "extraneous", since none of them can be expressed in terms of the others. All this needs to be carefully stated and proved of course, but this is the basic idea behind the introduction of linear independence.

Linear independence is closely connected with linear equations. For example, to see whether the vectors $\begin{bmatrix} -1 \\ 2 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 \\ -1 \\ -2 \\ 0 \end{bmatrix}$ are independent we consider how we can obtain $c_1\begin{bmatrix} -1 \\ 2 \\ 4 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 8 \\ -1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. This, in turn, is just the homogeneous system

$\begin{bmatrix} -1 & 2 & 8 \\ 2 & 1 & -1 \\ 4 & 2 & -2 \\ 3 & 2 & 0 \end{bmatrix} \bar{c} = \bar{0}$ where $\bar{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ is our vector of unknown coefficients. We know how to check for solutions: $\begin{bmatrix} -1 & 2 & 8 \\ 2 & 1 & -1 \\ 4 & 2 & -2 \\ 3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and we see that there is a nonzero solution. In fact this precise calculation tells us something else - we can express the third column (corresponding to the free variable) in terms of the first two: If we consider the problem $c_1 \begin{bmatrix} -1 \\ 2 \\ 4 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ -2 \\ 0 \end{bmatrix}$, the solution of that system using Gaussian elimination results in the exact same calculation:
$\left[\begin{array}{cccc} -1 & 2 & \vdots & 8 \\ 2 & 1 & \vdots & -1 \\ 4 & 2 & \vdots & -2 \\ 3 & 2 & \vdots & 0 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 0 & \vdots & -2 \\ 0 & 1 & \vdots & 3 \\ 0 & 0 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{array}\right]$ from which we can read off $c_1 = -2$, $c_2 = 3$ and
$(-2) \begin{bmatrix} -1 \\ 2 \\ 4 \\ 3 \end{bmatrix} + (3) \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ -2 \\ 0 \end{bmatrix}$

When vectors are linearly independent and we put them in a matrix $\begin{bmatrix} \bar{v}_1 & \bar{v}_2 & .. & \bar{v}_k \end{bmatrix}$ then the row reduced echelon form of that matrix corresonding to independent columns would have to be $\begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & 0 & & 1 \\ 0 & 0 & & 0 \end{bmatrix}$, i.e. the rank would have to be k, the number of columns.
(There would be $m - k$ rows of zeros, where m is the number of rows). To sum this up, we have:

The vectors $\bar{v}_1$, $\bar{v}_2, \ldots , \bar{v}_k$ are independent if and only if the matrix $\begin{bmatrix} \bar{v}_1 & \bar{v}_2 & \ldots & \bar{v}_k \end{bmatrix}$ has (full column) rank k, and if and only if the only solution of the homogeneous linear system $\begin{bmatrix} \bar{v}_1 & \bar{v}_2 & \ldots & \bar{v}_k \end{bmatrix} \bar{c} = \bar{0}$ is $\bar{c} = \bar{0}$.

From the other direction, if we reduce a matrix, we can say some things about its columns: If $A \sim R$, with R in row reduced echelon form, then

1) The set of columns in A that contain pivots in R is a linearly independent set.

2) Each column in A that does not contain a pivot can be expressed as a linear combination of the independent columns in A preceding it that contain a pivot in R. The coefficients in the linear combination are precisely the entries in the column that appear in R.

This is best illustrated with an example:

$$\begin{bmatrix} 1 & 0 & 1 & 2 & -1 & -2 & 1 & 4 \\ 0 & 1 & 2 & -3 & 1 & 1 & -1 & -1 \\ 2 & 0 & 2 & 4 & -1 & -3 & 1 & 6 \\ -1 & 1 & 1 & -5 & 2 & 3 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 & 0 & -1 & 0 & 2 \\ 0 & 1 & 2 & -3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

From this we can conclude:

1) Columns 1,2,5,7 are a set of 4 linearly independent vectors. For if we kept only these 4 columns and performed the same row operations, we would obtain

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 2 & 0 & -1 & 1 \\ -1 & 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2) Column 3 can be expressed as a linear combination of columns 1 and 2. For if we kept only columns 1,2,3 as an augmented matrix and performed the same row operations, we would obtain

$$\begin{bmatrix} 1 & 0 & \vdots & 1 \\ 0 & 1 & \vdots & 2 \\ 2 & 0 & \vdots & 2 \\ -1 & 1 & \vdots & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \vdots & 1 \\ 0 & 1 & \vdots & 2 \\ 0 & 0 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix} \text{ which tells us that } 1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

3) In a similar way, column 4 can be expressed as a linear combination of columns 1

and 2. For if we had only started with columns 1,2,4 we would arrive at:

$$\begin{bmatrix} 1 & 0 & \vdots & 2 \\ 0 & 1 & \vdots & -3 \\ 2 & 0 & \vdots & 4 \\ -1 & 1 & \vdots & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \vdots & 2 \\ 0 & 1 & \vdots & -3 \\ 0 & 0 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix} \text{ so that } 2\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \\ -5 \end{bmatrix}$$

4) Now including columns 1,2,5,6 in an augmented matrix, we would obtain (as seen above) $\begin{bmatrix} 1 & 0 & -1 & \vdots & -2 \\ 0 & 1 & 1 & \vdots & 1 \\ 2 & 0 & -1 & \vdots & -3 \\ -1 & 1 & 2 & \vdots & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \vdots & -1 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$ whence

$$(-1)\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \\ 3 \end{bmatrix}$$

5) Finally, column 8 can be expressed in terms of column 1,2,5,7 in a similar fashion.

Incidentally, these observations show us why row reduced echelon form is unique - the row reduced echelon form contains information about which columns are independent and which columns can be represented as linear combinations of the preceding independent columns, along with the (unique) coefficients required in those representations. Such information can be associated with only one row reduced echelon form.