Solutions   57

Math 251   Exam 3

1. Sketch, and label with their function value, representative level curves of the following functions:

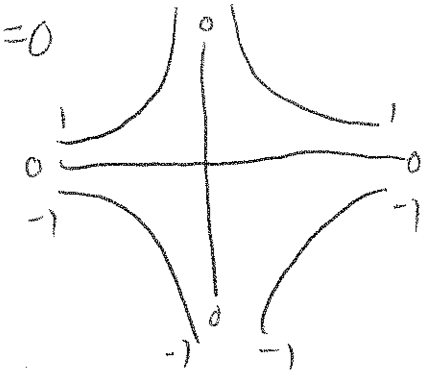
$f(x,y) = x^2y$ (be sure to include the $f = 0$ level curve curve)

3

$x^2y = 0$, $y = 0$ or $x = 0$

$x^2y = k$

$y = k/x^2$

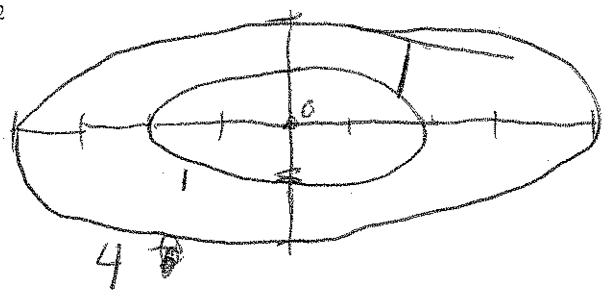


$f(x,y) = \frac{x^2}{4} + y^2$

3

$\frac{x^2}{4} + y^2 = k$

ellipses

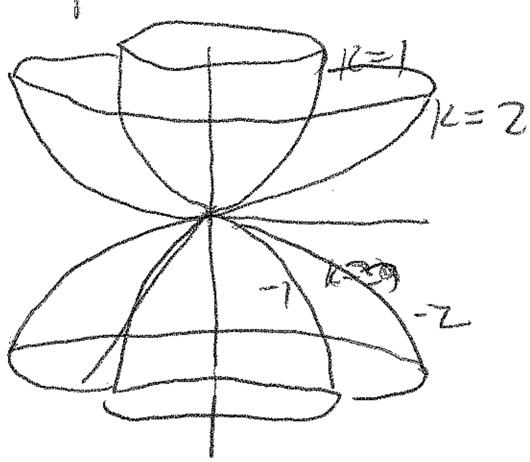


2. Sketch/describe the level surfaces of the function $f(x,y,z) = \frac{x^2+y^2}{z}$.

3

$\frac{x^2+y^2}{z} = k$, $z = \frac{1}{k}(x^2+y^2)$

elliptic paraboloids



3. Find the indicated partial derivatives:

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$f(x,y,z) = x^2y + x - y + \frac{1}{1+xz}$

$f_x = 2xy + 1 - \frac{z}{(1+xz)^2}$

$f_{xy} = 2x$

$f_{xz} = -\left[\frac{(1+xz)^2 \cdot 1 - z\, 2(1+xz)\, x}{(1+xz)^4}\right]$

$= \cancel{\frac{2zx - 1}{(1+xz)^2}} \quad \frac{2xz - (1+xz)}{(1+xz)^3} = \frac{xz-1}{(1+xz)^3}$

4. Find the linear approximation of $f(x,y) = \frac{9xy}{x^2-y^2}$ near $(x,y) = (-1,2)$

$f(-1,2) = \frac{-18}{1-4} = 6$

6

$f_x = \frac{(x^2-y^2)9y - 9xy \cdot 2x}{(x^2-y^2)^2} = 9\left[\frac{-3 \cdot 2 - 4}{9}\right] = -10$

$\boxed{f \approx 6 - 10(x+1) - 5(y-2)}$

$f_y = \frac{(x^2-y^2)9x - 9xy(-2y)}{(x^2-y^2)^2} = \frac{9}{9}\left[(-3)(-1) - 2 \cdot 4\right] = -5$

5. Find the equation of the plane tangent to the surface $x^2 + \ln(z^2 - 2y^2) = 1$ at the point $(1,2,3)$.

6

$f(x,y,z) = x^2 + \ln(z^2 - 2y^2)$

$\boxed{2(x-1) - 8(y-2) + 6(z-3) = 0}$

$f_x = 2x = 2$

$f_y = \frac{-4y}{z^2-2y^2} = -8$

$f_z = \frac{2z}{z^2-2y^2} = 6$

6. **Use the chain rule** to calculate the following derivatives:

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$\frac{dw}{dt}$ at $t = \frac{\pi}{4}$, where $w = 2x^2 + y^2 - z^2$, $x = \cos t$, $y = \sin t$, $z = t$; $t = \pi/4$, $x = \frac{1}{\sqrt{2}}$, $y = 1/\sqrt{2}$

$z = \pi/4$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= (4x)(-\sin t) + 2y\cos t - 2(z)\cdot 1$$

$$= -2 + 1 - \pi/2 = -1 - \pi/2$$

$\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at $(u,v) = (0,0)$ where $z = xy - x + y$, $x = ue^v$, $y = ve^u$

$(u,v) = (0,0)$, $(x,y) = (0,0)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (y-1)e^v + (x+1)ve^u$$

$$\boxed{\frac{\partial z}{\partial u} = -1}$$

$$\frac{\partial z}{\partial v} = (y-1)ue^v + (x+1)e^u$$

$$\boxed{\partial z/\partial v = 1}$$

7. Let $f(x,y,z) = xzy + xy - yz^2$.

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a) Calculate the directional derivative of at $(1,-1,1)$ in the direction $\langle -1,3,1\rangle$.

$$\vec{\nabla} f = \langle yz + y,\ xz + x - z^2,\ xy - 2yz\rangle$$

$$= \langle -2, 1, 1\rangle$$

$$D_{\vec{u}} f(p) = \frac{\langle -2,1,1\rangle \cdot \langle -1,3,1\rangle}{\sqrt{11}} = \frac{6}{\sqrt{11}}$$

b) At $(1,-1,1)$ in what direction is f increasing most rapidly, and what is that rate of increase per unit distance.

At $(1,-1,1)$ increasing most rapidly in direction $\langle -2,1,1\rangle$

with $D_{\vec{u}} f = \sqrt{6}$

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8. Locate and classify the critical points of the function

$f(x,y) = x^4 - 2x^2 - 16xy + 4y^2$

$f_x = 4x^3 - 4x - 16y = 0, \quad y = \frac{1}{4}(x^3 - x)$

$f_y = -16x + 8y = 0$

$y = 2x$

$\frac{1}{4}(x^3 - x) = 2x$

$x^3 - x = 8x$

$x^3 - 9x = 0$

$x(x^2 - 9) = 0$

$x = 0, \pm 3$

$f_{xx} = 12x^2 - 4$

$f_{yy} = 8$

$f_{xy} = -16$

(x,y)	Δ	
$(3,6)$	$(104)(8) - 16^2 > 0$	local min
$(-3,-6)$	$(104)(8) - 16^2 > 0$	local min
$(0,0)$	$-32 - 256 < 0$	saddle pt