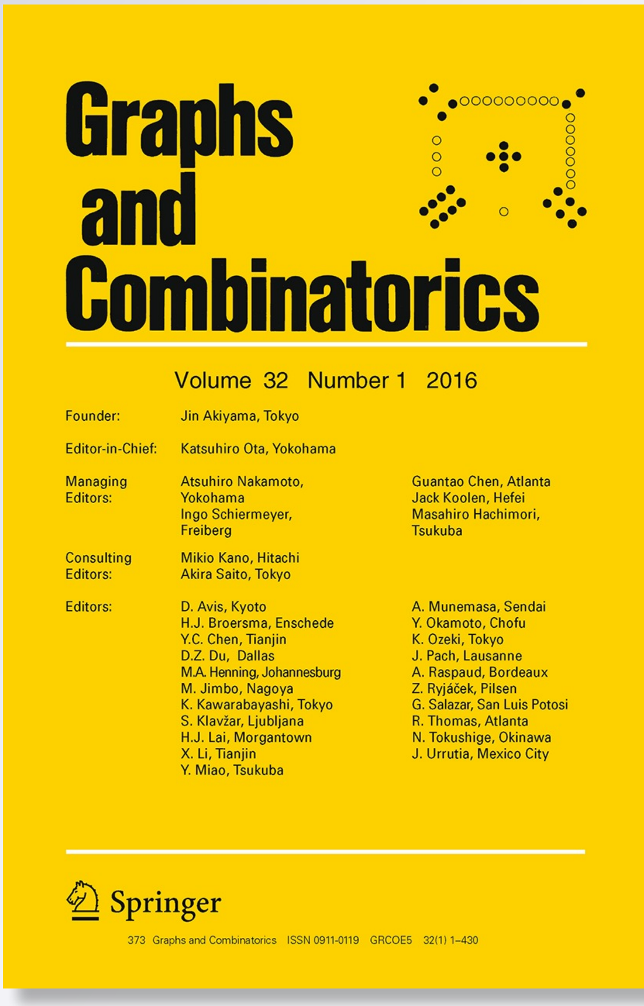
Hamilton Circuits and Essential Girth of Claw Free Graphs

Zhengke Miao, Xiaofeng Wang & Cun-Quan Zhang

Graphs and Combinatorics

ISSN 0911-0119
Volume 32
Number 1

Graphs and Combinatorics (2016)
32:311-321
DOI 10.1007/s00373-015-1559-9



Springer

Graphs and Combinatorics (2016) 32:311–321
DOI 10.1007/s00373-015-1559-9

ORIGINAL PAPER

Hamilton Circuits and Essential Girth of Claw Free Graphs

Zhengke Miao · Xiaofeng Wang · Cun-Quan Zhang

Received: 22 August 2014 / Revised: 7 January 2015 / Published online: 7 March 2015

Abstract Let G be a $K_{1,3}$-free graph. A circuit of G is essential if it contains a non-locally connected vertex v and passes through both components of $N(v)$. The essential girth of G, denoted by $g_e(G)$, is the length of a shortest essential circuit. It can be seen easily that, by Ryjáček closure operation, the essential girth of G is closely related to the girth of H where H is the Ryjáček closure of G and is a line graph. A generalized net, denoted by N_{i_1,i_2,i_3}, is a graph obtained from a triangle C_3 and three disjoint paths $P_{i_\mu+1}$ ($\mu = 1, 2, 3$), by identifying each vertex v_μ of $C_3 = v_1v_2v_3v_1$ with an end vertex of the path $P_{i_\mu+1}$, for every $\mu = 1, 2, 3$. In this paper, we prove that every 2-connected $\{K_{1,3}, N_{1,1,g_e(G)-4}\}$-free (and $\{K_{1,3}, N_{1,0,g_e(G)-3}\}$-free) graph G contains a Hamilton circuit. With the additional parameter g_e, these results extend some earlier theorems about Hamilton circuits in $\{K_{1,3}, N_{a,b,c}\}$-free graphs (for some small integers a, b and c).

Partially supported by an NSF-China Grant: NSFC 11171288 for Zhengke Miao, and partially supported by an NSF Grant DMS-1264800 and NSA Grants H98230-12-1-0233 and H98230-14-1-0154 for Cun-Quan Zhang.

Z. Miao
School of Mathematics and Statistics, Jiangsu Normal University,
Jiangsu 221116, China
e-mail: zkmiao@jsnu.edu.cn

X. Wang
Department of Mathematics and Actuarial Science, Indiana University Northwest,
Gary, IN 46408, USA
e-mail: wang287@iun.edu

C.-Q. Zhang (✉)
Department of Mathematics, West Virginia University, Morgantown,
WV 26506, USA
e-mail: cqzhang@math.wvu.edu

Keywords Claw-free graph · Net-free graph · Hamilton circuit · Forbidden pairs · Essential girth · Closure operation

1 Introduction

Let G be a graph. If a subgraph G' of G contains all edges $xy \in E(G)$ with $x, y \in V(G')$, then G' is called an induced subgraph of G. For a given graph H, we say that G is H-$free$ if G does not contain an induced subgraph isomorphic to H.

A claw is the complete bipartite graph $K_{1,3}$. A simple graph G is claw-free if it has no induced subgraph $K_{1,3}$.

Let P_i be the path on i vertices, and C_i the circuit on $i \geq 3$ vertices. We adopt the definition of generalized net in [4]. A generalized net, denoted by N_{i_1,i_2,i_3}, is a graph obtained from a triangle C_3 and three disjoint paths $P_{i_\mu+1}$ ($\mu = 1, 2, 3$), by identifying each vertex v_μ of $C_3 = v_1v_2v_3v_1$ with an end vertex of the path $P_{i_\mu+1}$, for every $\mu = 1, 2, 3$.

We call a graph G hamiltonian if it contains a Hamilton circuit, i.e., a circuit containing all its vertices. Regarding a graph to be hamiltonian, the following theorem is one of the earliest results in this subject of forbidden pairs.

Theorem 1.1 [6] *Let G be a* $\{K_{1,3}, N_{1,1,1}\}$*-free graph. If G is 2-connected, then G is hamiltonian.*

And it is followed by some other forbidden pairs.

Theorem 1.2 [3] *If G is a 2-connected* $\{K_{1,3}, P_6\}$*-free graph, then G is hamiltonian.*

Theorem 1.3 [10] *If G is a 2-connected* $\{K_{1,3}, N_{0,0,2}\}$*-free graph, then G is hamiltonian.*

Theorem 1.4 [1] *If G is a 2-connected* $\{K_{1,3}, N_{0,1,2}\}$*-free graph, then G is hamiltonian.*

Theorem 1.5 [7] *If G is a 2-connected* $\{K_{1,3}, N_{0,0,3}\}$*-free graph, then G is hamiltonian.*

Faudree and Gould [8] later refined this approach by the following classification theorem.

Theorem 1.6 [8] *Let G be a 2-connected,* $\{K_{1,3}, S\}$*-free graph. Then G is hamiltonian if S is one of* C_3, $N_{0,0,1}$, $N_{0,0,2}$, $N_{0,0,3}$, $N_{0,1,1}$, $N_{1,1,1}$, $N_{0,1,2}$ *(see Fig. 1).*

The following is one of the major open problems in this subject.

Conjecture 1.1 [14] *Every 4-connected claw-free graph is hamiltonian.*

There are other related results about forbidden pairs or local structures for Hamilton circuits in claw-free graphs regarding Conjecture 1.1, such as [2,5,12,15]. Lai et al. [13] and Fujisawa [9] recently showed some results for the forbidden pairs including a generalized net graph.

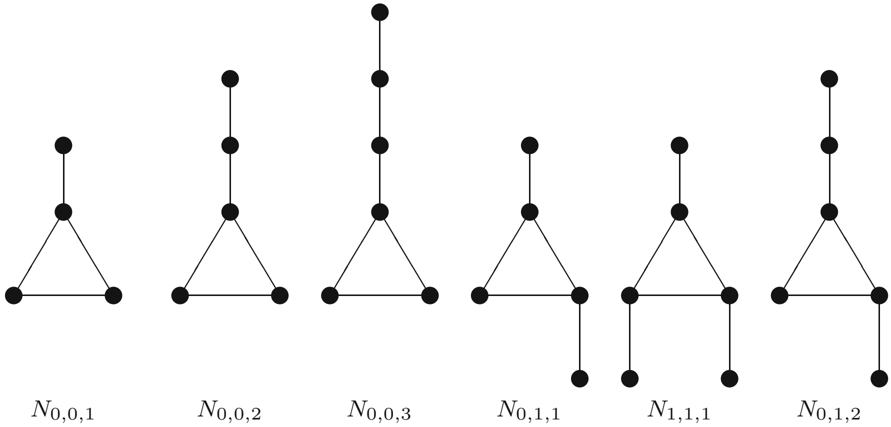


Fig. 1 $N_{0,0,1}$, $N_{0,0,2}$, $N_{0,0,3}$, $N_{0,1,1}$, $N_{1,1,1}$ and $N_{0,1,2}$

Theorem 1.7 [13] *If G is a 3-connected* $\{K_{1,3}, N_{0,0,8}\}$*-free graph, then G is hamiltonian.*

Theorem 1.8 [9] *If G is a 3-connected* $\{K_{1,3}, N_{0,0,9}\}$*-free graph, then G is hamiltonian unless G is the line graph of* Q^**, where* Q^* *is obtained from the Petersen graph by adding one pendent edge to each vertex.*

For a graph G and $v \in G$, denote the neighbor of v to be $N(v) = \{u \in V(G)$: u is adjacent to $v\}$.

Definition 1.1 For a claw free graph $G = (V, E)$, a vertex v is locally connected if the induced subgraph $G[N(v)]$ is connected. And denote $V_{lc} = \{v \in V(G) : v$ is locally connected$\}$, $V_{nlc} = \{v \in V(G) : v$ is not locally connected$\}$.

It is evident that if v is not locally connected, then the induced subgraph $G[N(v)]$ consists of two cliques Q_1^v, Q_2^v (see Lemma 2.2).

Definition 1.2 Let G be a claw-free graph. A circuit $C = v_1 \ldots v_r v_1$ of G is *essential* if it contains a non-locally connected vertex v_i and $v_{i-1}v_{i+1} \notin E(G)$.

Notice that the essential circuit C, with a non-locally connected vertex v_i, should pass through both clique components of $N(v_i)$ for otherwise there will be an edge between v_{i-1} and v_{i+1}.

Definition 1.3 The essential girth of a claw-free graph, denoted by $g_e(G)$, is the length of a shortest essential circuit.

Furthermore, it can be seen easily that, by Ryjáček closure operation [16], the essential girth of G is closely related to the girth of H where H is the line graph of the Ryjáček closure of G.

Here is the main theorem of this paper.

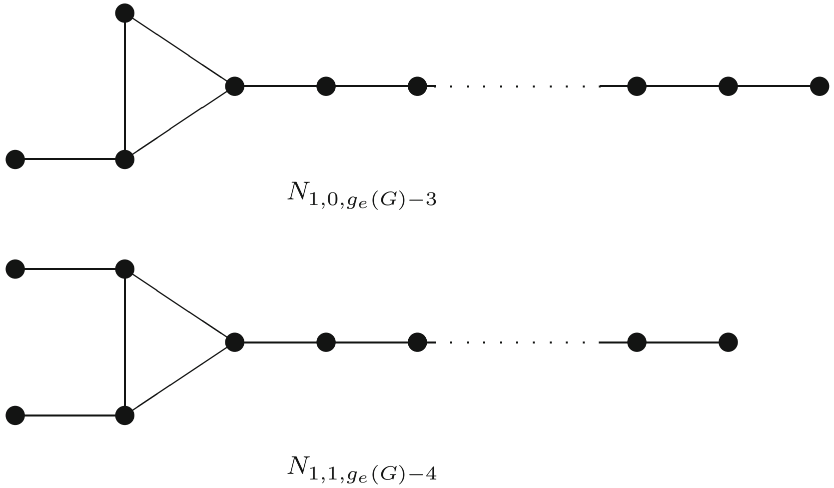


Fig. 2 $N_{1,0,g_e(G)-3}$ and $N_{1,1,g_e(G)-4}$

Theorem 1.9 (1) *If a 2-connected graph G is claw-free and $N_{1,1,g_e(G)-4}$-free, then G contains a Hamilton circuit.*

(2) *If a 2-connected graph G is claw-free and $N_{1,0,g_e(G)-3}$-free, then G contains a Hamilton circuit* (*see Fig.* 2).

With the additional parameter g_e, these results extend some earlier theorems about Hamilton circuits in $\{K_{1,3}, N_{a,b,c}\}$-free graphs (for some small integers a, b and c).

2 Preliminary Results and Lemmas

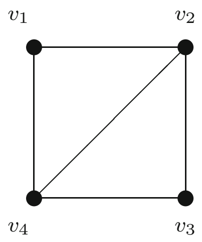
In this section, we present some early results and useful lemmas for the preparation of the proof of Theorem 1.9.

If H is a graph, then the *line graph* of H, denoted by $L(H)$, is the graph on vertex set $E(H)$ in which two vertices are adjacent if and only if their corresponding edges in H share an end vertex. A graph G is a line graph if it is isomorphic to $L(H)$ for some graph H. Note that line graphs are claw-free.

We say a subgraph is *even* if the degree of each vertex in the subgraph is even. And a *dominating connected even subgraph* of a graph H is a connected even subgraph such that every edge of H has at least one end vertex contained in the connected even subgraph. There is an intimate relationship between dominating connected even subgraph in H and Hamilton circuit in $L(H)$, a result due to Harary and Nash-Williams [11] that is known since the 1960s.

Lemma 2.1 [11] *Let H be a graph of size at least three. The line-graph $L(H)$ has a Hamilton circuit if and only if H has a dominating connected even subgraph.*

Fig. 3 $K_4 - \{e\}$



A closure operation is to add edges to turn $G[N(v)]$ into a complete graph for every locally connected vertex $v \in V(G)$ until it is impossible to add any more edges.

The following theorem due to Ryjáček [16] provides an opportunity to translate questions on hamiltonicity of claw-free graphs to questions on hamiltonicity of line graphs by using the concept of closure operation.

Theorem 2.1 [16] *Let G be a claw-free graph. Then* (1) *G has a Hamilton circuit if and only if the closure* $cl(G)$ *has a Hamilton circuit.* (2) *Furthermore,* $cl(G)$ *is the line graph* $L(H)$ *for some graph* H.

The following is a well-known folklore result for claw-free graphs.

Lemma 2.2 *Let G be a claw-free graph. For* $v \in V_{nlc}$*, the induced subgraph* $G[N(v)]$ *consists of two disjoint cliques* Q_1^v, Q_2^v*. Furthermore, if the non-locally connected vertex* v *is contained in a vertex-cut* T *with components* R_1, R_2 *of* $G - T$*, then* $Q_i^v \subseteq R_i \cup T$ *for* $i = 1, 2$.

Lemma 2.3 *Let* v *be a non-locally connected vertex of a claw-free graph* G *with components* Q_1^v, Q_2^v *of* $G[N(v)]$*. Let* $C = vx_1, \ldots, x_r v$ *be a circuit of* G *containing* v *and* $x_1 \in Q_1^v, x_r \in Q_2^v$*. If each* x_i *is locally connected in* G *for every* $i \in \{2, \ldots, r-1\}$*, then* v *is locally connected in the closure* $cl(G)$.

Proof We first claim that, *for each* $i, j \in \{1, \ldots, r\}$*, there is an edge* $x_i x_j$ *in the closure* $cl(G)$.

We prove this claim by induction on $|j - i|$.

The claim is true for $|j - i| = 1$.

Now consider $|j - i| \geq 2$. Let $1 \leq i < j \leq r$. The vertex x_{i+1} remains locally connected in $cl(G)$ since the closure only adds edges. Notice that $x_j \in N_{cl(G)}(x_{i+1})$ by the induction hypothesis. Then both $x_i, x_j \in N_{cl(G)}(x_{i+1})$, so the edge $x_i x_j$ must appear in $cl(G)$. This finishes the proof of the claim.

By the above claim, $x_1 x_r \in E(cl(G))$, which joins Q_1^v, Q_2^v. Thus, $N_{cl(G)}(v)$ induces a connected subgraph in $cl(G)$, so v is locally connected in the closure $cl(G)$. □

Lemma 2.4 *Let G be a claw-free graph. If Q is an induced subgraph of G isomorphic to* $K_4 - \{e\}$ *(see Fig. 3) where* $V(K_4) = \{v_1, \ldots, v_4\}$ *and* $e = v_1 v_3$*, then both* v_2, v_4 *are locally connected vertices.*

Proof If v_2 is not locally connected, by Lemma 2.2, vertices v_1, v_3, v_4 are contained in the same clique, which contradicts that v_1v_3 is a missing edge in G. Similarly v_4 is also locally connected. □

Lemma 2.5 *Let G be a claw-free graph with the essential girth $g_e(G) \geq 5$, if $C = v_1v_2v_3v_4v_1$ is an induced 4-circuit of G, then all v_i are locally connected vertices for $1 \leq i \leq 4$.*

Proof Suppose that v_1 is not locally connected. By the definition of the essential girth $g_e(G)$ and the assumption that $g_e(G) \geq 5$, both vertices v_2, v_4 are in the same component Q_i^v for some $i \in \{1, 2\}$. And by Lemma 2.2, $v_2v_4 \in E(G)$ which contradicts that C is an induced circuit, so v_1 is locally connected. Similar arguments can be applied to v_2, v_3, and v_4. □

Lemma 2.6 *Let $T = \{u, v\}$ be a cut of a 2-connected claw-free graph G, then*

(i) *either both u and v are non-locally connected in G*
(ii) *or both u and v are locally connected, and adjacent to each other.*

Proof Suppose that at least one of $\{u, v\}$, say u, is locally connected in G. Let R_1, R_2 be two components of $G - T$. Since u is locally connected in G, let P be a shortest path contained in $G[N(u)]$ joining $R_1 \cap N(u)$ and $R_2 \cap N(u)$. It is obvious that $P = x_1vx_2$ where $x_i \in R_i \cap N(u)$ $(i = 1, 2)$. The subgraph $G[u, v, x_1, x_2]$ is isomorphic to $K_4 - \{x_1x_2\}$. By Lemma 2.4, v is also locally connected. □

3 Proof of the Main Theorem

In this section, we present the proof of Theorem 1.9.

We prove this result by contradiction. Let G be a counterexample to Theorem 1.9 with $|V(G)| + |E(G)|$ as small as possible. This means G is a 2-connected, claw-free and either $N_{1,1,g_e(G)-4}$-free or $N_{1,0,g_e(G)-3}$-free graph. For $g_e(G) = 4, 5$, the theorem is already proved in Theorem 1.6, so for the rest part of the proof, we will assume that $(*)$ $g_e(G) \geq 6$.

If every vertex in the closure $cl(G)$ is locally connected, then $cl(G)$ is a complete graph. By Ryjáček's closure Theorem (Theorem 2.1), $cl(G)$ has a Hamilton cycle.

Thus, we may assume that there exist non-locally connected vertices in $cl(G)$.

Claim 3.1 *For every non-locally connected vertex v of $cl(G)$, v is contained in a 2-vertex cut of G.*

Proof Let v be a non-locally connected vertex of $cl(G)$ (and non-locally connected in G, as well) and supposed that v is not contained in any 2-vertex cut of G.

By Lemma 2.2, let Q_1^v, Q_2^v be components of $G[N(v)]$. Without loss of generality, let $|Q_2^v| \leq |Q_1^v|$. Let F be the set of edges between v and Q_2^v, that is $F = [\{v\}, Q_2^v]$. And let

$$W_i = \{x \in V(G) : \ distance_{(G-F)}(x, v) = i\}.$$

Let r be the smallest integer that $Q_2^v \cap W_r \neq \emptyset$.

Since $G - v$ is 2-connected, there exist two chordless paths $P_1 = x_1 \ldots x_{r-1}x_r$, $P_2 = y_1 \ldots y_{r-1}y_r$ such that

(1) $x_i, y_i \in W_i$ for $1 \le i \le r$,
(2) $x_i \neq y_i$ for $2 \le i \le r-1$,
(3) $x_r \in Q_2^v$.

Notice that by the definition of the essential girth $g_e(G)$, we have $r \ge g_e(G) - 1$. □

Subclaim 3.1.1 *If $x_i y_{i+1} \in E(G)$ [or, $y_i x_{i+1} \in E(G)$], for some i: $1 \le i \le r-1$, then $x_i y_i, x_{i+1} y_{i+1} \in E(G)$.*

Proof If $x_{i+1} y_{i+1} \notin E(G)$, then $G[x_i, x_{i-1}, y_{i+1}, x_{i+1}]$ is a claw by the definition of the sets W_i. Hence, $x_{i+1} y_{i+1} \in E(G)$. Symmetrically, $x_i y_i \in E(G)$. □

Without confusion, let $v = x_0 = y_0$.

For an integer $i \in \{0, \ldots, r-1\}$, the pair $\{i, i+1\}$ is called a *non-diagonal pair* if $x_i \neq y_i$, $x_{i+1} \neq y_{i+1}$ and both $x_i y_{i+1}, y_i x_{i+1} \notin E(G)$; the pair $\{i, i+1\}$ is called a *diagonal pair* if both $x_i y_{i+1}, y_i x_{i+1} \in E(G)$ or $x_i = y_i$ or $x_{i+1} = y_{i+1}$. Note that there may be pairs that are neither.

Subclaim 3.1.2 *For each $i \in \{0, \ldots, r-2\}$,*

(1) *if $x_{i+1} \neq y_{i+1}$, then $x_{i+1} y_{i+1} \in E(G)$, and*
(2) *if $\{i, i+1\}$ is a non-diagonal pair, then $\{i+1, i+2\}$ is a diagonal pair.*

Proof This proof is by induction on i.

For $i = 0$, there is nothing to prove for statement (1) since x_1, y_1 are in the clique Q_1^v. Also, there is nothing to prove for statement (2) either, since both $vx_1, vy_1 \in E(G)$ where $v = x_0 = y_0$. In the case that $x_1 = y_1$, to avoid a claw centered at $x_1 = y_1$, we get $x_2 y_2 \in E(G)$.

Assume that there is an integer J ($1 \le J \le r-2$) such that both statements hold for every i with $0 \le i \le J-1$.

The following is the proof of the statements (1) and (2) for $i = J$.

Suppose that $x_{i+1} y_{i+1} \notin E(G)$. By Subclaim 3.1.1, the assumption that $x_{i+1} y_{i+1} \notin E(G)$ implies that both $\{i, i+1\}$ and $\{i+1, i+2\}$ are non-diagonal pairs. The pair $\{i-1, i\}$ cannot be non-diagonal pair, for otherwise, by inductive hypotheses of statement (2), $\{i, i+1\}$ cannot be non-diagonal either.

In summary, $x_{i+1} y_{i+1} \notin E(G)$, and, both $\{i, i+1\}$ and $\{i+1, i+2\}$ are non-diagonal pairs. And $\{i-1, i\}$ is not a non-diagonal pair. Without loss of generality, let $x_{i-1} y_i \in E(G)$. Thus, the subgraph induced by vertices

$$\{v, x_1, \ldots, x_r\} - \{x_{i+2}\} + \{y_i, y_{i+1}\}$$

contains a generalized net $N_{1,1,g_e(G)-4}$; and the subgraph induced by vertices

$$\{v, x_1, \ldots, x_r\} - \{x_{i+1}\} + \{y_i, y_{i+1}\}$$

contains a generalized net $N_{1,0,g_e(G)-3}$. This is a contradiction and proves the statement (1).

For the statement (2), assume that $\{i, i+1\}$ is a non-diagonal pair and assume that $x_{i+2} \neq y_{i+2}$. Then $y_{i+1}x_{i+2} \in E(G)$ for otherwise, $G[x_{i+1}, x_i, y_{i+1}, x_{i+2}]$ is a claw. Similarly, $y_{i+2}x_{i+1} \in E(G)$. This proves the statement (2). □

Subclaim 3.1.3 $\{x_i, y_i\}$ *are locally connected vertices of G for each* $i = 2, \ldots, r-1$.

Proof Consider $i \in \{2, \ldots, r-1\}$. By Lemma 2.5, we may assume that neither $\{i-1, i\}$ nor $\{i, i+1\}$ is a non-diagonal pair. Hence, let $u' \in \{x_{i-1}, y_{i-1}\} \cap N(x_i) \cap N(y_i)$ and $u'' \in \{x_{i+1}, y_{i+1}\} \cap N(x_i) \cap N(y_i)$. The subgraph of G induced by $\{u', u'', x_i, y_i\}$ is $K_4 - e$. By Lemma 2.4, both x_i, y_i are locally connected. □

Proof Thus, we have a circuit $C = vx_1x_2 \ldots x_r v$ passing through both Q_1^v, Q_2^v such that every vertex x_i is locally connected for every $i \in \{2, \ldots, r-1\}$. By Lemma 2.3, v is locally connected in $cl(G)$. And this completes the proof of Claim 3.1. □

Thus, every non-locally connected vertex of $cl(G)$ is contained in a 2-vertex cut of G.

Let $T = \{u, v\}$ be a 2-vertex cut separating the graph G into two components $Q_1(u, v)$ and $Q_2(u, v)$, let $G_i(u, v) = Q_i(u, v) \cup T$ for each $i = \{1, 2\}$. [That is, $G_1(u, v) \cup G_2(u, v) = G$ and $V(G_1(u, v)) \cap V(G_2(u, v)) = T$]. We say $G_i(u, v)$ is *path-trivial* for i $\in \{1, 2\}$ if $G_i(u, v)$ is either a chordless path with ends at v and u or a chordless circuit containing the edge vu.

Claim 3.2 *At least one of* $\{G_1(u, v), G_2(u, v)\}$ *is path-trivial.*

Proof Let $P_i(u, v)$ be a shortest path of $G_i(u, v)$ joining v and u, for $i = 1, 2$. Let $G_i^*(u, v) = G_i(u, v) \cup P_j(u, v)$, for every $\{i, j\} = \{1, 2\}$.

Prove by contradiction. Assume that $G_j(u, v) \neq P_j(u, v)$, for every $j \in \{1, 2\}$. Since G is a smallest counterexample, each $G_i^*(u, v)$ contains a Hamilton circuit $C_i(u, v)$. Note that the circuit $C_i(u, v)$ contains the non-trivial path $P_j(u, v)$ for every $\{i, j\} = \{1, 2\}$. Joining $C_1(u, v)$ and $C_2(u, v)$, we have a Hamilton circuit in G.

Thus, we may assume that $G_2(u, v) = P_2(u, v)$. That is, $G_2(u, v)$ is a chordless path or a chordless circuit containing the edge vu. □

Now, for every 2-vertex cut $T = \{u, v\}$ of G, at least one of $\{G_1(u, v), G_2(u, v)\}$ is path-trivial (suppose $G_2(u, v)$). And furthermore, both v and u are non-locally connected in both G and $cl(G)$ due to Lemma 2.6.

Define $\mathcal{T}$ be the set of all 2-vertex cuts of G.

(∗∗) *Choose* $T = \{w, w'\} \in \mathcal{T}$ *such that the path trivial part* $|V(Q_2(w, w'))|$ *is as large as possible.*

Note that every essential circuit passing through w (or w', as well) must contain $G_2(w, w')$.

(∗ ∗ ∗) *Let C be a shortest essential circuit passing through* w *and* w',

where $C = v_1 \ldots v_r \ldots v_s v_1$ with $v_1 = w$, $v_r = w'$, $v_{r+1} \ldots v_s = Q_2(w, w')$, and $v_r v_{r+1} \ldots v_s v_1 = G_2(w, w')$. It is easy to see that $r \geq 2$, $s \geq r+1$.

Define $Y_i = \{y \notin V(C) : distance_G(y, V(C)) = i\}$. By Claim 3.2, for each $i \geq 1$, $Y_i \subseteq Q_1(v_1, v_r)$.

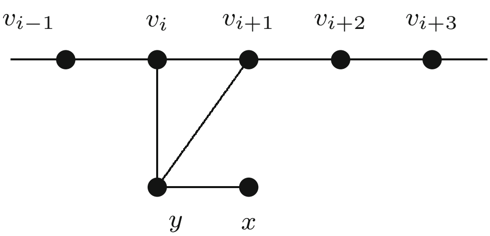


Fig. 4 $N(y) \cap V(C) = \{v_i, v_{i+1}\}$

Claim 3.3 *For each* $y \in Y_1$,

(i) *if* y *is not locally connected, then* $N(y) \cap V(C) = \{v_i, v_{i+1}\}$ *for some* $i \in \{1, \ldots, r-1\}$.
(ii) *if* y *is locally connected, then* $\{v_i, v_{i+1}\} \subseteq N(y) \cap V(C) \subseteq \{v_i, v_{i+1}, v_{i+2}\}$ *for some* $i \in \{1, \ldots, r-1\}$.

Proof Since $G_2(v_1, v_r) = v_r \ldots v_s v_1$ is path trivial, $N(y) \cap V(C) \subseteq \{v_1, \ldots, v_r\}$. Choose $v_i \in N(y) \cap V(C)$ with i as small as possible.

Since C is a shortest essential circuit passing through v_1 and v_r, we have each vertex $v_j \notin N(y)$, where $i + 3 \leq j \leq r$, for otherwise, $v_1 \ldots v_i y v_j \ldots v_r \ldots v_s v_1$ is shorter than C.

If $v_{i+1} \notin N(y)$, then $G[v_i, y, v_{i-1}, v_{i+1}]$ is a claw. Hence, both $v_i, v_{i+1} \in N(y)$.

If $v_{i+2} \in N(y)$ and y is not locally connected, then, by Lemma 2.2, C has a chord $v_i v_{i+2}$. This contradicts that C is chordless. □

Claim 3.4 $Y_2 \cup Y_3 \cup Y_4 \cdots = \emptyset$

Proof We prove this result by contradiction. Let $x \in Y_2$ and xyv_i be a shortest path joining x and $V(C)$ and choose v_i such that the subscript i is as small as possible. Note that $N(y) \cap V(C) \subseteq \{v_1, v_2, \ldots, v_r\}$ since $\{v_{r+1}, \ldots, v_s\} = V(Q_2(w, w'))$ (see Fig. 4).

By Claim 3.3, $\{v_i, v_{i+1}\} \subseteq N(y) \cap V(C) \subseteq \{v_i, v_{i+1}, v_{i+2}\}$ for some $i \in \{1, \ldots, r-1\}$. Furthermore, if $v_{i+2} \in N(y) \cap V(C)$, then $G[y, x, v_i, v_{i+2}]$ is a claw. Therefore, $N(y) \cap V(C) = \{v_i, v_{i+1}\}$. Hence, it is easy to see that induced subgraphs $N_{1,1,g_e(G)-4}$ and $N_{1,0,g_e(G)-3}$ are contained in the subgraph $G[x, y, V(C)]$ which contradicts the assumption. □

Claim 3.5 *Every vertex in* Y_1 *is locally connected in* G.

Proof Assume not. Choose $z \in Y_1$ and $v_i \in V(C)$ such that z is not locally connected in G and $v_i z \in E(G)$ and i is as small as possible. By Claim 3.3, $N(z) \cap V(C) = \{v_i, v_{i+1}\}$ and $1 \leq i \leq r-1$. Here, assume that $\{v_i, v_{i+1}\} \subseteq Q_1^z$. By Lemma 2.2 and Claim 3.3, let $z' \in Q_2^z$ and $z' \notin V(C)$ (see Fig. 5).

By Claim 3.4, $z' \in Y_1$ and let $z'v_k \in E(G)$ where $v_k \in V(C)$ with k as large as possible. Since $g_e(G) \geq 6$, the length of the essential circuit $zv_{i+1}v_{i+2} \ldots v_{k-1}v_k z' z$ is at least 6. Thus, we obtain another essential circuit $zz'v_k v_{k+1} \ldots v_i z$ of $w = v_1$ that is of length shorter than C, a contradiction. □

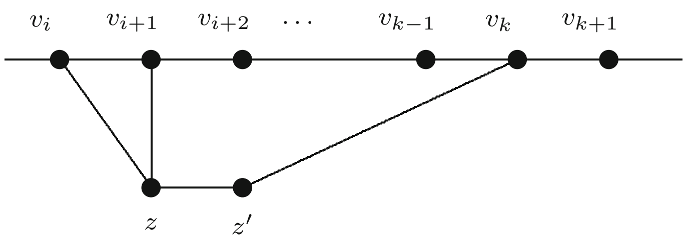


Fig. 5 Every vertex in Y_1 is locally connected in G

Claim 3.6 *Furthermore, every vertex v of the component $Q_1(v_1, v_r)$ is locally connected in $cl(G)$.*

Proof Prove by contradiction. Let v be a non-locally connected vertex in $cl(G)$. By Claim 3.5, $v \notin Y_1$. That is, $v = v_i \in V(C)$. Here, $1 < i < r$. Choose v_i such that i is as small as possible.

By Claim 3.1, v_i is contained in a 2-vertex cut $T' = \{v_i, x\}$.

Case 1. $x \in V(C)$.

We may assume that $x = v_h$ with h as small as possible. That is, by the choice of i, we have $i+1 \leq h \leq r$. Let $Q_1(v_i, v_h)$ and $Q_2(v_i, v_h)$ be components of $G - \{v_i, v_h\}$. By Claim 3.2, let $Q_2(v_i, v_h)$ be path trivial. If $Q_2(v_i, v_h)$ contains $Q_2(v_1, v_r)$, then it contradicts $(**)$. Since every vertex in the path trivial part $Q_2(v_i, v_h)$ is non-locally connected, by Claim 3.5, it is not in Y_1. That is, $Q_2(v_i, v_h)$ is the segment $v_i \ldots v_h$ of C and therefore, $i+1 < h \leq r$.

By $(**)$, let P^* be a shortest path of $G - Q_2(v_1, v_r) - Q_2(v_i, v_h)$ joining $\{v_1, \ldots, v_i\}$ and $\{v_h, \ldots, v_r\}$. By Claim 3.4, $P^* - V(C)$ has only two vertices z_1, z_2 where z_1 is adjacent to v_a with $1 \leq a \leq i$ while z_2 is adjacent to v_b with $h \leq b \leq r$. By $(*)$ and Claim 2.2, the essential circuit $v_a \ldots v_b z_2 z_1 v_a$ is of length at least 6. And by $(***)$, $|\{v_a, \ldots, v_b\}| = 4$.

Reroute the circuit C by replacing the segment $v_a \ldots v_b$ with the path P^*, and apply Claim 3.5 to both vertices of $\{v_{a+1}, v_{b-1}\}$. This contradicts that each vertex of $Q_2(v_i, v_h) = v_{i+1} \ldots v_{h-1}$ is non-locally connected in G.

Case 2. $x \notin V(C)$.

By Claim 3.4, $x \in Y_1$. Thus, $V(C) - v_i \subseteq Q_1(v_i, x)$. By Claim 3.5, x is locally connected in G which contradicts Lemma 2.6 that x must be non-locally connected in G as v_i is non-locally connected. □

So, every vertex of G, except for $V(G_2(v_1, v_r)) = \{v_r, \ldots, v_s, v_1\}$, is locally connected and $Q_2(v_1, v_r)$ is a path attaching v_1, v_r. Then, in the closure $cl(G)$, the subgraph induced by $V(Q_1(v_1, v_r))$ is a complete graph, and the subgraph induced by $V(Q_2(v_1, v_r))$ remains as the path attaching v_1, v_r or a circuit containing the edge $v_1 v_r$. It is easy to see that $cl(G)$ has a Hamilton circuit and, by Theorem 2.1, so is G. It contradicts that G is a counterexample and, therefore, completes the proof of the Theorem 1.9.

References

1. Bedrossian, P.: Forbidden subgraph and minimum degree conditions for hamiltonicity. Ph.D. thesis, Memphis State University (1991)
2. Broersma, H.J., Kriesell, M., Ryjáček, Z.: On factors of 4-connected claw-free graphs. J. Graph Theory **37**, 125–136 (2001)
3. Broersma, H.J., Veldman, H.J.: Restrictions on induced subgraphs ensuring Hamiltonicity or pancyclicity of $K_{1,3}$-free graphs. In: Bondendiek, R. (ed.) Contemporary Methods in Graph Theory, pp. 181–194. BI-Wiss-Verlag, Mannheim (1990)
4. Brousek, J., Faudree, R.J., Ryjáček, Z.: A note on hamiltonicity of generalized net-free graphs of large diameter. Discrete Math. **251**, 77–85 (2002)
5. Chiba S., Fujisawa, J.: Hamiltonicity of 3-connected generalized bull-free line graphs. http://www.rs.kagu.tus.ac.jp/chiba/forbidden-subgraph(B_%7Bs,t%7D).pdf
6. Duffus, D., Gould, R.J., Jacobson, M.S.: Forbidden Subgraphs and the Hamiltonian Theme, The Theory and Applications of Graphs, Kalamazoo, Michigan, 1980. Wiley, New York (1981)
7. Faudree, R.J., Gould, R.J., Ryjáček, Z., Schiermejer, I.: Forbidden subgraphs and pancyclicity. Congr. Numer. **109**, 13–32 (1995)
8. Faudree, R.J., Gould, R.J.: Characterizing forbidden pairs for hamiltonian properties. Discrete Math. **173**, 45–60 (1997)
9. Fujiwasa, J.: Forbidden Subgraphs for Hamiltonicity of 3-connected claw-free graphs. J. Graph Theory **73**, 146–160 (2013)
10. Gould, R.J., Jacobson, M.S.: Forbidden subgraphs and hamiltonian properties of graphs. Discrete Math. **42**, 189–196 (1982)
11. Haray, F., Nash-Williams, C.S.J.A.: On Eulerian and Hamiltonian graphs and line graphs. Can. Math. Bull. **8**, 701–710 (1965)
12. Kaiser, T., Li, M., Ryjáček, Z., Xiong, L.: Hourglass and Hamilton cycle in 4-connected claw-free graphs. J. Graph Theory **48**, 267–276 (2005)
13. Lai, H.-J., Xiong, L., Yan, H., Yan, J.: Every 3-connected claw-free Z_8-free graph is Hamiltonian. J. Graph Theory **64**, 1–11 (2010)
14. Matthews, H.M., Summer, D.P.: hamiltonian results in $K_{1,3}$-free graphs. J. Graph Theory **8**, 139–146 (1984)
15. Pfender, Florian: Hamiltonicity and forbidden subgraphs in 4-connected graphs. J. Graph Theory **49**, 262–272 (2005)
16. Ryjáček, Z.: On a closure concept in claw-free graphs. J. Comb. Theory Ser. B **70**, 217–224 (1997)