Graphs and Combinatorics (1997) 13: 189–196

Graphs and Combinatorics

Decomposition of the Flow Polynomial

K. Sekine[1] and C.Q. Zhang[2]*

[1] Department of Information Science, University of Tokyo, Tokyo 113, Japan
[2] Department of Mathematics, West Virginia University, Morgantown, WV 26506-63 10, USA

Abstract. The flow polynomials denote the number of nowhere-zero flows on graphs, and are related to the well-known Tutte polynomials and chromatic polynomials. We will show the decomposition of the flow polynomials by edge-cuts and vertex-cuts of size 2 or 3. Moreover by using this decomposition, we will consider what kind of graphs have the same flow polynomials. Another application of the decomposition results is that if a bridgeless graph G does not admit a nowhere-zero k-flow and G has a small vertex- or edge-cut, then a proper bridgeless subgraph of G (a graph minor) does not admit a nowhere-zero k-flow either.

1. Introduction

Nowhere-zero flow problems, which were introduced by Tutte (1954, [7]) are a subject of combinatorics [2, 6]. For example, for planar graphs the problems are equivalent to face-coloring problems. They are also related to the problems of cycle covers (families of cycles which cover all edges) such as cycle double cover conjecture, cycle cover of short total length [1], and so on.

The flow polynomials are one of invariants for a given graph. In this paper, we will consider the decomposition of the flow polynomials. For 2-edge-cuts and 3-edge-cuts, the flow polynomial of a given graph can be represented by the product of the flow polynomials of two graphs which are obtained by the decomposition. We also consider the decomposition of the flow polynomials by 2-vertex-cuts and 3-vertex-cuts.

Let $G = (V, E)$ be a graph with a vertex set $V(G)$ and an edge set $E(G)$. The definition of a nowhere-zero t-flow is as follows.

Definition 1. *Suppose that H is any Abelian group with the order t and ω is any orientation of the edges for a given graph G. A* nowhere-zero flow *is defined as a*

* This research was partially supported by the National Science Foundation under Grant DMS-9306379 and by the National Security Agency under Grant MDA904-96-1-0014.

mapping φ: $E(G) \to H\backslash\{0\}$ *(where* 0 *is the zero of* H*) such that for each vertex* v *of* G,

$$\sum_{\delta^+(v)} \varphi(e) - \sum_{\delta^-(v)} \varphi(e) = 0,$$

where $\delta^+(v)$ *and* $\delta^-(v)$ *denote, respectively, the edges of* G *which are oriented out of and into* v *in the orientation* ω. *If* φ *takes all its values in* $H\backslash\{0\}$, *then it is called a* nowhere-zero t-flow.

Let $F(G; t)$ denote the number of nowhere-zero t-flows on G. As we will see in the contraction-deletion formula, $F(G; t)$ is a polynomial in t, and will be called the *flow polynomial* of G.

Let $P(G; t)$ denote the number of vertex colorings of G with t colors such that no two adjacent vertices have the same color. This is the well-known *chromatic polynomial* of G. Both the flow polynomial and the chromatic polynomial are special cases of the 2-variable polynomial; the Tutte polynomial.

The flow polynomial can be evaluated by the following contraction-deletion formula [7]. For a bridgeless graph G,

$$F(G;t) = F(G/e;t) - F(G\backslash e;t) \qquad \text{if } e \text{ is not a loop}$$

$$F(G;t) = (t-1)\cdot F(G\backslash e;t) \qquad \text{if } e \text{ is a loop.}$$

$$\begin{cases} F(I;t) = 0 & \text{where } I \text{ and } L \text{ denote an isthmus} \\ F(L;t) = t-1 & \text{and a loop, respectively.} \end{cases}$$

Here, $G\backslash e$ denotes the graph obtained from G by *deleting* the edge e and G/e the graph obtained by removing e and *contracting* the endpoints of e to a single vertex. Furthermore, a *loop* is an edge connecting the same vertex, and an *isthmus* (or a *bridge*) is an edge whose removal increases the connected components of the graph by one.

This formula determines $F(G; t)$ uniquely, that is, it does not depend on the ordering of deletions and contractions. Moreover it does not depend on the orientation ω of G nor the structure of the group H.

2. Decomposition of the Flow Polynomial

For a connected graph G, the i-edge-cut of G is defined as an edge set $A \subseteq E(G)$ whose removal disconnects G and $|A| = i$. Similarly the j-vertex-cut of G is defined as a vertex set $B \subseteq V(G)$ whose removal disconnects G and $|B| = j$.

Trivially, if G consists of connected components G_1 and G_2, $F(G;t) = F(G_1;t)F(G_2;t)$. Then we consider only connected graphs in this section.

2.1. Edge-cut Decompositions

It is known that if G has a bridge (i.e. 1-edge-cut) then $F(G; t) = 0$ for any t (see [7]). For 2-edge-cuts and 3-edge-cuts, there exists some simple decomposition. For a planar graph this decomposition is the dual case of the following well-known result for the chromatic polynomial. However this decomposition can be conducted for a non-planar graph.

If $G = G_1 \cup G_2$ where $G_1 \cap G_2 = K_n$, then

$$P(G;t) = \frac{P(G_1;t)P(G_2;t)}{P(K_n;t)}.$$

Here, the *union* $G_1 \cup G_2$ of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph with the vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. The *intersection* $G_1 \cap G_2$ is defined similarly. See Read [4] for the proof.

A loopless graph with two vertices and h edges is denoted by K_2^h. Hereafter $G \backslash A$ denotes the graph obtained from G by deleting all edges in A.

Lemma 1. *The only bridgeless bipartite graph with 2 or 3 edges is K_2^2 or K_2^3. Consequently, the only bipartite graph with 2 or 3 edges and with non-zero flow polynomial is K_2^2 or K_2^3.*

Theorem 1. *Let G be a graph with an edge-cut A: $2 \le |A| \le 3$. Let G_1 and G_2 be the graphs obtained from G by contracting one component of $G \backslash A$. Then*

$$F(G;t) = \frac{F(G_1;t)F(G_2;t)}{F(K_2^h;t)}$$

where $h = |A|$.

Proof. Prove by induction on the number of edges. It is obvious for graphs with at most h edges (by Lemma 1). Assume that the theorem is true for each graph with at most $m - 1$ edges. Consider any bridgeless graph G with m edges. Let $e \in E(G)\backslash A$, say, $e \in E(G_1)\backslash A$. By the contraction-deletion formula, we have that

Left hand side:

$$F(G;t) = F(G/\{e\};t) - F(G\backslash\{e\};t);$$

Right hand side:

$$\frac{F(G_1;t)F(G_2;t)}{F(K_2^h;t)} = \frac{(F(G_1/\{e\};t) - F(G_1\backslash\{e\};t))F(G_2;t)}{F(K_2^h;t)}$$

$$= \frac{F(G_1/\{e\};t)F(G_2;t)}{F(K_2^h;t)} - \frac{F(G_1\backslash\{e\};t)F(G_2;t)}{F(K_2^h;t)}$$

By the induction hypothesis,

$$F(G/\{e\};t) = \frac{F(G_1/\{e\};t)F(G_2;t)}{F(K_2^h;t)}$$

and

$$F(G\backslash\{e\};t) = \frac{F(G_1\backslash\{e\};t)F(G_2;t)}{F(K_2^h;t)}.$$

Therefore the result holds for all graphs. □

Note that, in the processing of edge deletion, the term $F(G_1 \backslash \{e\}; t)$ will be eliminated (a zero polynomial) if $G_1 \backslash \{e\}$ has a bridge. This guarantees that the reduction ends at the unique graph K_2^h.

However we do not have a similar decomposition for edge-cut A of size at least 4 because Lemma 1 does not hold for $|A| \geq 4$.

Corollary 1. *Let G be a bridgeless graph with an edge-cut A of size 2 or 3. Let G_1 and G_2 be the graphs obtained from G by contracting one component of $G \backslash A$. If G does not admit a nowhere-zero t-flow, then either G_1 or G_2 does not admit a nowhere-zero t-flow.*

This corollary is a strengthening of a lemma by Seymour for 2-edge-cuts ([5], or see [2]).

2.2. Vertex-cut Decompositions

Proposition 1. *Let G be a graph and v be a cut-vertex of G separating G into two parts G_1 and G_2 (that is, $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \{v\}$). Then*

$$F(G; t) = F(G_1; t)F(G_2; t).$$

Theorem 2. *Let G be a graph and $B = \{x_1, x_2\}$ be a 2-vertex-cut separating G into two parts G_1 and G_2. Let H_i be the graph obtained from G_i by adding a new edge joining x_1 and x_2, for both $i = 1, 2$. Then*

$$F(G; t) = \frac{F(H_1; t)F(H_2; t)}{F(K_2^2; t)} + F(G_1; t)F(G_2; t).$$

Proof. Let H be a graph obtained from G_1 and G_2 by adding 2 new edges joining $x_j \in V(G_1)$ and $x_j \in V(G_2)$ for each $j = 1, 2$. By applying the contraction-deletion formula for these two edges, we obtain

$$F(H; t) = F(G; t) - F(G_1; t)F(G_2; t).$$

On the other hand, by Theorem 1,

$$F(H; t) = \frac{F(H_1; t)F(H_2; t)}{F(K_2^2; t)}. \qquad \square$$

Theorem 3. *Let G be a graph and $B = \{x_1, x_2, x_3\}$ be a 3-vertex-cut separating G into two parts G_1 and G_2. Let H_i be the graph obtained from G_i by adding a new vertex v_i and adding 3 new edges joining v_i and each x_j, for both $i = 1, 2$ and each $j \in \{1, 2, 3\}$. Let $J_i^{\{a,b\}}$ be the graph obtained from G_i by adding a new edge joining x_a and x_b for both $i = 1, 2$ and $a, b \in \{1, 2, 3\}$ and $a \neq b$. Then*

$$\begin{aligned} F(G; t) = &\frac{F(H_1; t)F(H_2; t)}{F(K_2^3; t)} \\ &+ \frac{F(J_1^{\{1,2\}}; t)F(J_2^{\{1,2\}}; t) + F(J_1^{\{1,3\}}; t)F(J_2^{\{1,3\}}; t) + F(J_1^{\{2,3\}}; t)F(J_2^{\{2,3\}}; t)}{F(K_2^2; t)} \\ &+ F(G_1; t)F(G_2; t). \end{aligned}$$

Proof. Let H be a graph obtained from G_1 and G_2 by adding 3 new edges joining $x_j \in V(G_1)$ and $x_j \in V(G_2)$ for each $j \in \{1, 2, 3\}$. Similar to Theorem 2, first we apply the contraction-deletion formula for these three edges. Then the formula is obtained by using Theorem 1 and Theorem 2. □

The following results are immediate corollaries for a smallest graph admitting no nowhere-zero k-flows.

Corollary 2. *Let G be a graph and $B = \{x_1, x_2\}$ be a 2-vertex-cut separating G into two parts G_1 and G_2. Let H_i be the graph obtained from G_i by adding a new edge joining x_1 and x_2, for both $i = 1, 2$. If G does not admit a nowhere-zero k-flow, then one of $\{G_1, G_2\}$ and one of $\{H_1, H_2\}$ do not admit a nowhere-zero k-flow for integer k.*

Corollary 3. *Let G be a graph and $B = \{x_1, x_2, x_3\}$ be a 3-vertex-cut separating G into two parts G_1 and G_2. Let H_i be the graph obtained from G_i by adding a new vertex v_i and adding 3 new edges joining v_i and each x_j, for both $i = 1, 2$ and each $j \in \{1, 2, 3\}$. Let $J_i^{\{a,b\}}$ be the graph obtained from G_i by adding a new edge joining x_a and x_b for both $i = 1, 2$ and $a, b \in \{1, 2, 3\}$ and $a \neq b$. If G does not admit a nowhere-zero k-flow, then one of $\{G_1, G_2\}$, one of $\{H_1, H_2\}$ and one of $\{J_1^{\{a,b\}}, J_2^{\{a,b\}}\}$ for each pair $a, b \in \{1, 2, 3\}$ $(a \neq b)$ do not admit a nowhere-zero k-flow for integer k.*

3. Graphs with the Same Flow Polynomials

Since any graph with a bridge must have a zero flow polynomial, we consider only bridgeless graphs in this section.

We notice that subdividing an edge of a graph does not affect the flow polynomial of the graph. The underlying graph of a graph G, denoted by $\overline{G}$, is the graph obtained from G by replacing all subdivided edges with single edges. The following observation is obvious.

Fact 1. *For two graphs G_1 and G_2, if $\overline{G_1} = \overline{G_2}$ then $F(G_1; t) = F(G_2; t)$.*

Here, we are interested in the following problem.

Problem 1. *If $F(G_1; t) = F(G_2; t)$, is $\overline{G_1} = \overline{G_2}$? If not, is there any identical substructural information that we may have from both graphs?*

3.1. Graph Isomorphism and Flow Polynomials

Definition 2. The $\Delta \to Y$ operation *is an operation of a graph that contracts a triangle to a vertex.* The $Y \to \Delta$ operation *is an operation of a graph that expands a vertex with degree* 3 *to a triangle. In the both operations three vertices of the triangle have degree* 3.

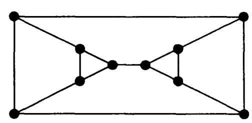
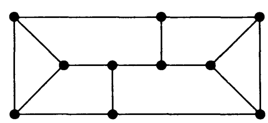
By Theorem 1, we have the following fact.

Fact 2. *Let G be a graph obtained from another graph H by a $Y \to \Delta$ operation. Then $F(G;t) = (t-3)F(H;t)$.*

By Fact 2 and applying a series of $\Delta \to Y$ operations and $Y \to \Delta$ operations for a graph, we have the following immediate result.

Fact 3. *Let G and H be two graphs with $|E(\overline{G})| = |E(\overline{H})|$. If two graphs obtained from $\overline{G}$ and $\overline{H}$ respectively by series of $\Delta \leftrightarrow Y$ operations are isomorphic, then $F(G;t) = F(H;t)$.*

Example 1. *The following graphs G and H can be transformed to isomorphic graphs by applying $\Delta \to Y$ operations for two triangles of G and H respectively. Therefore, they are not isomorphic graphs but have the same flow polynomial by Fact* 3.

G  H 

Example 1 shows that if two bridgeless graphs have the same flow polynomials, their underlying graphs may not necessarily be isomorphic. Generally, we have the following two corollaries by Theorem 1.

An edge-cut A of a graph G is *non-trivial* if no component of $G \backslash A$ is a single vertex. Define an operation for a graph G as follows. Let A be a non-trivial edge-cut of G of size $\mu \leq 3$. Let G_1, G_2 be graphs obtained from G by contracting one component of $G \backslash A$. Then, we say that G is decomposed to graphs G_1 and G_2 by a *μ-edge-cut decomposition.* G_1 and G_2 are called factors of G. Recursively apply this operation to each factor (a factor of a factor of G is also considered as a factor of G) until each factor has no non-trivial edge-cut of size 2 or 3.

Corollary 4. *Let G be a graph such that G is decomposed to $\{G_1, \ldots, G_{r+1}\}$ by a series of r_3 3-edge-cut decompositions and r_2 2-edge-cut decompositions ($r = r_2 + r_3$). Then*

$$F(G;t) = \frac{\prod_{i=1}^{r+1} F(G_i;t)}{(t-2)^{r_3}(t-1)^r}.$$

Note that $F(K_2^2;t) = t-1$ and $F(K_2^3;t) = (t-1)(t-2)$.

Corollary 5. *Let G and H be two graphs such that G is decomposed to $\{G_1, \cdots, G_{r+1}\}$ by a series of r_3 3-edge-cut decompositions and a series of r_2 2-edge-cut decompositions, and H is decomposed to $\{H_1, \cdots, H_{r+1}\}$ by a series of r_3 3-edge-cut decompositions and a series of r_2 2-edge-cut decompositions ($r = r_2 + r_3$). If there is a bijection ϕ: $\{G_1, \cdots, G_{r+1}\} \mapsto \{H_1, \cdots, H_{r+1}\}$ such that $\overline{\phi(G_i)} = \overline{G_i}$ for every i, then $F(G;t) = F(H;t)$.*

For example, since both graphs in Example 1 are decomposed to four complete graphs K_4 by three 3-edge-cut decompositions, they have the same flow polynomial:

$$F(G;t) = F(H;t) = (t-1)(t-2)(t-3)^4.$$

3.2. Cycle Spaces and Flow Polynomials

How about graphs without non-trivial 2- or 3-edge-cuts? With a similar method as above for edge-cuts, we can construct pairs of non-isomorphic graphs with small vertex-cuts and the same flow polynomials. For 1- and 2-vertex-cuts they correspond to 2-isomorphism as follows.

Two graphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ are *2-isomorphic* if there is a bijection $\psi: E_1 \to E_2$ such that the edge set $E'_1 \subseteq E_1$ is a cycle in H_1 if and only if $\psi(E'_1)$ is a cycle in H_2. It is obvious that the property of graphical isomorphic implies the property of 2-isomorphic.

By Whitney's 2-isomorphism theorem [3], 2-isomorphic graphs have the same Tutte polynomials [8]. Then we have the following proposition.

Proposition 2. *If H_1 and H_2 are 2-isomorphic, then $F(H_1; t) = F(H_2; t)$.*

From this proposition, we have the following two facts.

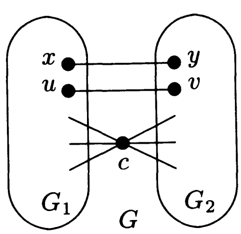
Fact 4. *Let G be a graph and v be a cut-vertex of G. Let H be a graph consists of two connected components which obtained by separating G at v. Then $F(G; t) = F(H; t)$.*

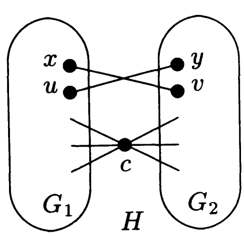
Fact 5. *Suppose that the graph G is obtained from the disjoint graphs G_1 and G_2 by identifying the vertices x of G_1 and y of G_2, and identifying the vertices u of G_1 and v of G_2. Suppose that H is a graph obtained from G_1 and G_2 by identifying x and v, u and y. Then $F(G; t) = F(H; t)$.*

Note that the properties of 2-isomorphism and graphic isomorphism are equivalent for 3-connected graphs (by Theorem 1, [8] p. 83). Hence, a pair of non-isomorphic 3-connected graphs must have different cycle spaces (that is, not 2-isomorphic). Here we introduce another construction method which constructs a pair of 3-connected non-isomorphic graphs G_1 and G_2 with $F(G_1; t) = F(G_2; t)$. Note that the graphs constructed here have 3-vertex-cuts, while the edge-connectivity could be arbitrary high.

Example 2. *Let c be a vertex of a graph G such that $G \backslash \{c\}$ has a 2-edge-cut $\{xy, uv\}$, where $\{x, u\}$ and $\{y, v\}$ are in the two distinct components of $G \backslash \{c\}$. Let the vertex c separate the graph $G \backslash \{xy, uv\}$ to two parts G_1 and G_2.*

Then let H be the graph obtained from $G \backslash \{xy, uv\}$ by adding edges $\{xv, uy\}$.





By applying Theorem 3 on the 3-vertex-cut $\{x, u, c\}$ of G and H, we have that

$$F(G; t) = F(H; t).$$

Now, the graphs G_1 and G_2 in Problem 1 should be at least 3-connected, and furthermore, for each vertex c of G_i, $G_i \backslash \{c\}$ is still 3-edge-connected.

Problem 2. *Let $\mathscr{G}$ be the set of all 4-connected graphs. Define an equivalent relation $\equiv_{fp}$ in $\mathscr{G}$ as follows. For a pair of G_1, $G_2 \in \mathscr{G}$, $G_1 \equiv_{fp} G_2$ if $F(G_1;t) = F(G_2;t)$. If $G_1 \equiv_{fp} G_2$, is there any identical substructural information between G_1 and G_2? Or, is it possible that $G_1 \equiv_{fp} G_2$ if and only if G_1 and G_2 are isomorphic?*

Acknowledgment. The first author would like to thank Hiroshi Imai who read through the earlier draft of this paper and gave her helpful comments.

References

1. Fan, G.: Integer flows and cycle covers. J. Comb. Theory Ser. B **54**, 113–122 (1992)
2. Jaeger, F.: Nowhere-zero flow problems. In: L.W. Beineke et al.: Selected topics in graph theory **3** (chapter 4, pp. 71–95) Academic-Press 1988
3. Oxley, J.G.: Matroid theory, Oxford Science Publications 1992.
4. Read, R.C.: An introduction to chromatic polynomials. J. Comb. Theory **4**, 52–71 (1968)
5. Seymour, P.D.: Nowhere-zero 6-flows. J. Comb. Theory Ser. B **30**, 130–135 (1981)
6. Seymour, P.D.: Nowhere-zero flow. In: R. Graham et al.: Handbook of combinatorics, Elsevier Science Publishers 1993
7. Tutte, W.T.: A contribution to the theory of chromatic polynomials. Canad. J. Math. **6**, 80–91 (1954)
8. Welsh, D.J.A.: Matroid theory (London Math. Soc., vol. 8), Academic Press London 1976

Received: September 8, 1995
Revised: July 3, 1996