Flows, flow-pair covers and cycle double covers

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1. Introduction

Assume that $C$ is a dominating circuit of a cubic graph $G$. A matching $M$ of $G$ is bipartizing with respect to $C$ if $M \cap E(C) = \emptyset$, $V(G) - V(C) \subseteq V(M)$ and either the cubic graph homeomorphic to $G - E(M)$ is bipartite or $G - E(M)$ is 2-regular. The concept of bipartizing matching was introduced in [3,5]. If $G$ contains a pair of edge-disjoint bipartizing matchings with respect to a dominating circuit $C$, then it was proved in [3] that $G$ has a 5-cycle double cover and was proved in [5] that $G$ admits a nowhere-zero 5-flow. Applying an observation by Tutte [15] (Lemma 2.1) that a cubic graph admits a nowhere-zero 3-flow if and only if the graph is bipartite, one can see that the statement that a cubic graph contains a pair of edge-disjoint bipartizing matchings with respect to a dominating circuit implies the existence of a pair of 3-flows with a special arrangement of the flow values in the graph. This relation inspires a further investigation of flow coverings of graphs in this paper.

The concept of flow-pair covering, introduced in many publications, has been used as one of the major techniques in the studies of integer flows and cycle covers. A pair of integer flows $(D, f_1)$ and $(D, f_2)$ is a cover of a graph $G$ if $\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$. The following lemma addresses the relationship between flow covers and nowhere-zero flows.

**Lemma 1.1.** A graph $G$ admits a nowhere-zero hk-flow if and only if $G$ admits an h-flow $(D, f_1)$ and a k-flow $(D, f_2)$ such that $\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$.

The "if" part of Lemma 1.1 was applied in the proofs of many celebrated results in the integer flow theory [7,11]. If a graph is covered by two flows, then it admits a nowhere-zero flow as their "product" (by Lemma 1.1). If a graph $G$ is covered by two flows in a certain special arrangement, then it is possible that $G$ may admit a nowhere-zero flow stronger/better...
Lemma 1.1. He proved that if a cubic graph $G$ with a dominating circuit $C$ has a pair of 3-flows $((D, f_1), (D, f_2))$ with $E_{f_1 \text{odd}} = E_{f_2 \text{odd}} = E(C)$ covering $G$, then $G$ admits a nowhere-zero 5-flow and has a circuit double cover. These pioneer results by Fleischner \([4,5]\) are to be generalized in this paper for graphs without requiring the existence of a dominating circuit and 3-regularity.

Some famous theorems in integer flow theory were proved by applying Lemma 1.1: the 6-flow theorem \([11]\) is proved via the product of a pair of a 3-flow and a 2-flow, the 4-flow theorem for 4-edge-connected graphs \([7]\) is proved via the product of a pair of 2-flows, and the 8-flow theorem \([7]\) is proved via the product of three 2-flows.

From all those approaches, one might propose that the famous 5-flow conjecture by Tutte \([16]\) could be approached by covering a bridgeless graph with a pair of a 2-flow and a $\frac{5}{2}$-flow which is defined as a circular flow, see \([6,20]\) for definition). However, this approach does not work for cubic graphs since the support of a $\frac{5}{2}$-flow must be odd-5-edge-connected. A result in this paper raises a new hope that the flow-pair covering approach introduced by Jaeger and Seymour \([7,11]\) may still work for the 5-flow conjecture although 5 is a prime number.

2. Notations and useful lemmas

For notations not defined here see \([1]\).

A circuit is a connected 2-regular graph, while a cycle is a graph with even degree for every vertex. An edge $e$ is a bridge of graph $G$ if the removal of $e$ increases the number of components. A graph $G$ is odd-(2$k$ + 1)-edge-connected if the size of every odd edge cut is at least $2k + 1$.

Let $G$ be a graph such that the degree of every vertex is either 2 or 3. The cubic graph isomorphic to $G$ is denoted by $\overline{G}$ and is called the underlying graph of $G$.

Let $Z$ be the set of all integers. Let $G = (V, E)$ be a graph and let $k$ be a positive integer. An ordered pair $(D, \phi)$ is called an integer $k$-flow of $G$ if $D = (V, A)$ is an orientation of $G$ and $\phi : A \mapsto Z$ is an assignment of flow values such that for every vertex $v$,

$$\sum_{e \in E^+(v)} \phi(e) = \sum_{e \in E^-(v)} \phi(e)$$

and

$$|\phi(e)| < k \quad \text{for all } e \in E(G)$$

where $E^+(v)$ (resp. $E^-(v)$) is the set of all edges of $G$ under the orientation $D$ with tail $v$ (resp. head $v$). We say that $(D, \phi)$ is a nowhere-zero flow if $\phi(e) \neq 0$ for all $e \in E(G)$. This concept was introduced by Tutte \([16]\), and the theory of nowhere-zero flows generalizes map-coloring theorems and conjectures for planar graphs to general graphs. Major open problems in this area are Tutte's celebrated 3-, 4-, and 5-flow conjectures. Interested readers are referred to \([8,13]\) for the main ideas of this subject and to \([18,19]\) for in-depth accounts.

Let $G = (V, E)$ be a graph and let $k$ be a positive integer. An ordered pair $(D, \phi)$ is called a modular $k$-flow of $G$ if $D = (V, A)$ is an orientation of $G$ and $\phi : A \mapsto Z$ is an assignment of flow values such that for every vertex $v$,

$$\sum_{e \in E^+(v)} \phi(e) \equiv \sum_{e \in E^-(v)} \phi(e) \pmod{k}.$$

We say that $(D, \phi)$ is nowhere-zero if $\phi(e) \not\equiv 0 \pmod{k}$ for all $e \in E(G)$.

For a flow $(D, f)$ of $G$, the support of $f$ is the set $\text{supp}(f) = \{e \in E(G) : f(e) \neq 0\}$. Denote $E_{f=\text{odd}} = f^{-1}(t)$, $E_{f=\text{even}} = f^{-1}(0)$, $E_{f=\text{odd}} = \{e \in E(G) : f(e) \equiv 1 \pmod{2}\}$, $E_{f=\text{even}} = \{e \in E(G) : f(e) \equiv 0 \pmod{2}\}$, and $E_{f=\text{even}, f \neq 0} = E_{f=\text{even}} \setminus \text{supp}(f)$.

A family $\mathcal{C}$ of cycles of a graph $G$ is a cycle cover of $G$ if $E(G) = \bigcup_{C \in \mathcal{C}} E(C)$. A cycle cover $\mathcal{C}$ is a $k$-cycle cover if $|\mathcal{C}| = k$. A cycle cover $\mathcal{C}$ is a double cover of $G$ if every edge of $G$ is contained in precisely two members of $\mathcal{C}$ \([10,14]\). A cycle double cover $\mathcal{C}$ is a $k$-cycle double cover if $|\mathcal{C}| \leq k$.

A $k$-cycle double cover $\{C_1, \ldots, C_k\}$ of $G$ is orientable if each $C_i$ has an Eulerian orientation such that each edge $e$ of $G$ is covered by two cycles $C_j$ and $C_l$ with opposite directions.

The following are some fundamental lemmas in the theory of integer flow, which are to be used frequently in the remaining part of the paper.

**Lemma 2.1** (Tutte \([15]\)). A cubic graph admits a nowhere-zero 3-flow if and only if it is bipartite.

**Lemma 2.2** (Tutte \([17]\)). A graph admits a nowhere-zero 2-flow if and only if it is a cycle.

**Lemma 2.3** (Tutte \([17]\)). Let $(D, f)$ be an integer flow of $G$. Then $E_{f=\text{odd}}$ induces a cycle of $G$.

**Lemma 2.4** (Tutte \([15]\)). If a graph $G$ admits a modular $k$-flow $(D, f)$, then $G$ admits an integer $k$-flow $(D, f')$ such that

$$f(e) \equiv f'(e) \pmod{k} \quad \text{for all } e \in E(G).$$

**Lemma 2.5** (\([15,12]\)). Let $H$ be a subgraph of $G$. Then $G$ admits a nowhere-zero 4-flow $(D, f)$ with $E_{f=\text{even}} = E(H)$ if and only if $G$ has a 2-cycle cover with the edges of $H$ covered twice and all other edges covered once.
3. Flow-pair coverings

**Definition 3.1.** Let $G$ be a graph and $h, k$ be two integers. Assume that $(D, f_1)$ is an integer $h$-flow of $G$ and $(D, f_2)$ is an integer $k$-flow of $G$. If $\supp(f_1) \cup \supp(f_2) = E(G)$ and $E_{f_1=\text{odd}} = E_{f_2=\text{odd}}$, then $\{(D, f_1), (D, f_2)\}$ is called an $(h, k)$-flow parity-pair-cover of $G$.

The distribution of edges with weights of different parity of an $(h, k)$-flow parity-pair-cover is illustrated in the following chart-like figure.

$$
\begin{array}{c|c|c}
\text{E}(G) & \text{even} & \text{odd} \\
\hline
f_1 : & \text{odd} & \text{even} \\
\hline
f_2 : & \text{even} & \text{odd} \\
\end{array}
$$

**Definition 3.2.** Let $G$ be a graph and $h, k$ be two integers. Assume that $(D, f_1)$ is an integer $h$-flow of $G$ and $(D, f_2)$ is an integer $k$-flow of $G$. If

$$
\supp(f_1) \cup \supp(f_2) = E(G)
$$

and

$$
E_{f_1=\text{even}, f_1 \neq 0} \subseteq E_{f_2=\text{odd}}, E_{f_2=\text{even}, f_2 \neq 0} \subseteq E_{f_1=\text{even}},
$$

then $\{(D, f_1), (D, f_2)\}$ is called an $(h, k)$-flow even-disjoint-pair-cover of $G$. (That is, $E(G)$ has a decomposition $\{A, B, C\}$ where $A = E_{f_1=\text{even}, f_1 \neq 0}$, $B = E_{f_2=\text{even}, f_2 \neq 0}$, $C = E_{f_1=\text{odd}} \cup E_{f_2=\text{even}}$.)

The distribution of edges with weights of different parity of an $(h, k)$-flow even-disjoint-pair-cover is illustrated in the following chart-like figure.

$$
\begin{array}{c|c|c|c}
\text{E}(G) & \text{even, } \neq 0 & \text{odd} & \text{0} \\
\hline
f_1 : & \text{odd} & \text{even, } \neq 0 & \text{0} \\
\hline
f_2 : & \text{0} & \text{odd} & \text{even, } \neq 0 \\
\end{array}
$$

**Definition 3.3.** Let $G$ be a graph and $h, k$ be two integers. Assume that $(D, f_1)$ is an integer $h$-flow of $G$ and $(D, f_2)$ is an integer $k$-flow of $G$. If $\{(D, f_1), (D, f_2)\}$ is both an $(h, k)$-flow parity-pair-cover and an $(h, k)$-flow even-disjoint-pair-cover, then $\{(D, f_1), (D, f_2)\}$ is called an $(h, k)$-flow strong parity-pair-cover of $G$. (That is, the pair of integer flows $\{(D, f_1), (D, f_2)\}$ satisfies the descriptions of both Definitions 3.1 and 3.2.)

Most theorems in next sections are generalizations of some earlier results of Fleischner [4,5] for cubic graphs with respect to dominating circuits.

4. Nowhere-zero flows and flow covers

**Theorem 4.1.** Let $G$ be a graph. Then $G$ admits a nowhere-zero 6-flow if and only if $G$ admits a $(4, 3)$-flow parity-pair-cover.

**Proof.** $\Leftarrow$: Assume that $\{(D, f_1), (D, f_2)\}$ is a $(4, 3)$-flow parity-pair-cover of $G$. Table 1 verifies that $(D, \frac{3h}{2} + \frac{k}{2})$ is a nowhere-zero 6-flow of $G$.

$\Rightarrow$: Assume that $(D, f_3)$ is a nowhere-zero 6-flow of $G$. Let $(D, f_4)$ be an integer 2-flow of $G$ with $\supp(f_4) = E_{f_3=\text{odd}}$.

<table>
<thead>
<tr>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$\frac{3h}{2} + \frac{k}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>±2</td>
<td>±1</td>
</tr>
<tr>
<td>±2</td>
<td>0</td>
<td>±3</td>
</tr>
<tr>
<td>±2</td>
<td>±2</td>
<td>±2 or ±4</td>
</tr>
<tr>
<td>±1</td>
<td>±1</td>
<td>±2 or ±1</td>
</tr>
<tr>
<td>±3</td>
<td>±1</td>
<td>±5 or ±4</td>
</tr>
</tbody>
</table>

Table 1

By definition, \((D, f_3 + f_4 \over 2)\) is a modular 3-flow of \(G\). Applying Lemma 2.4, let \((D, f_5)\) be an integer 3-flow of \(G\) with 
\[f_5 \equiv f_3 + f_4 \pmod{3} \]

Note that 
\[\text{supp}(f_4) \cup \text{supp}(f_5) = E(G)\]
since the transition from a modular \(k\)-flow to a \(k\)-flow yields the same support.

Let \((D, f_6)\) be an integer 2-flow of \(G\) with \(\text{supp}(f_6) = E f_{\text{odd}}\). Then \((D, f_7 = f_6 + 2f_4)\) is an integer 4-flow of \(G\) with 
\[\text{supp}(f_7) \cup \text{supp}(f_5) = E(G)\]
and
\[E f_{\text{odd}} = \text{supp}(f_6) = E f_{\text{odd}}.\]

By a similar method, we can also prove that a graph admits a nowhere-zero 8-flow if and only if the graph admits a \((4, 4)\)-flow parity-pair-cover. Since it is known that every bridgeless graph admits a nowhere-zero 6-flow \([11]\), we omit the corresponding proof for 8-flow.

**Theorem 4.2.** Let \(G\) be a graph. Then \(G\) admits a nowhere-zero 4-flow if and only if \(G\) admits a \((3, 3)\)-flow strong parity-pair-cover.

**Proof.** \(\Rightarrow:\) Let \((D, f)\) be a nowhere-zero 4-flow. Let \((D, f_1)\) be an integer 2-flow with 
\[\text{supp}(f_1) = E f_{\text{odd}}.\]
Then
\[\left( D, f_2 = \frac{f + f_1}{2} \right)\]
is a 3-flow. Let \((D, f_3)\) be an integer 2-flow of \(G\) with 
\[\text{supp}(f_3) = E f_{\text{odd}}.\]
Then
\[\left\{ \left( D, \frac{f_1 + f_3}{2} \right), \left( D, \frac{f_1 - f_3}{2} \right) \right\}\]
is a \((3, 3)\)-flow strong parity-pair-cover of \(G\).

\(\Leftarrow:\) Assume that \(\{(D, f'), (D, f'')\}\) is a \((3, 3)\)-flow strong parity-pair-cover of the graph \(G\). Then \((D, \frac{3f' + f''}{2})\) is a nowhere-zero 4-flow of \(G\). See Table 2.

**Table 2**

<table>
<thead>
<tr>
<th>(f')</th>
<th>(f'')</th>
<th>(\frac{3f' + f''}{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>±2</td>
<td>±1</td>
</tr>
<tr>
<td>±2</td>
<td>0</td>
<td>±3</td>
</tr>
<tr>
<td>±1</td>
<td>±1</td>
<td>±2 or ±1</td>
</tr>
</tbody>
</table>

**Theorem 4.3.** Let \(G\) be a graph. If \(G\) admits a \((3, 3)\)-flow parity-pair-cover then \(G\) admits a nowhere-zero 5-flow.

**Proof.** Assume that \(\{(D, f_1), (D, f_2)\}\) is a \((3, 3)\)-flow parity-pair-cover of \(G\). Then \((D, \frac{3f_1 + f_2}{2})\) is a nowhere-zero 5-flow of \(G\). See Table 3.

**Table 3**

<table>
<thead>
<tr>
<th>(f_1)</th>
<th>(f_2)</th>
<th>(\frac{3f_1 + f_2}{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>±2</td>
<td>±1</td>
</tr>
<tr>
<td>±2</td>
<td>0</td>
<td>±3</td>
</tr>
<tr>
<td>±2</td>
<td>±2</td>
<td>±2 or ±4</td>
</tr>
<tr>
<td>±1</td>
<td>±1</td>
<td>±2 or ±1</td>
</tr>
</tbody>
</table>

In contrast to Theorems 4.1 and 4.3 is not an “if and only if” statement: the other direction is proposed as a conjecture (Conjecture 8.1).

Furthermore, one part of both Theorems 4.1 and 4.3 can be further generalized as follows:
In general, if $G$ admits an $(h, 3)$-flow parity-pair-cover, then $G$ admits a nowhere-zero $j$-flow where

\[ j = \frac{3h + 1}{2} \quad \text{if } h \text{ is odd}, \]

and \[ j = \frac{3h}{2} \quad \text{if } h \text{ is even}. \]

This general result is not posed as a theorem because we already know the existence of a nowhere-zero 6-flow for any bridgeless graph [11].

5. Flow parity-pair-covers and even-disjoint-pair-covers

**Theorem 5.1.** Let $G$ be a graph. Then $G$ admits a $(3, 3)$-flow parity-pair-cover $\{(D, f_1), (D, f_2)\}$ if and only if $G$ admits a $(3, 3)$-flow even-disjoint-pair-cover $\{(D, f'), (D, f'')\}$. Furthermore,

\[ E_{f_1=\pm 1} = E_{f_2=\pm 1} = E_{f'=1} \Delta E_{f''=1}. \]

**Proof.** $\Rightarrow$: Assume that $G$ admits a $(3, 3)$-flow parity-pair-cover $\{(D, f_1), (D, f_2)\}$. Then

\[ \{(D, f') = (D, \frac{f_1 + f_2}{2}), (D, f'') = (D, \frac{f_1 - f_2}{2})\} \]

is a $(3, 3)$-flow even-disjoint-pair-cover of $G$: for each $e \in E_{f_1=\pm 1} = E_{f_2=\pm 1}$, either $f'(e) = 1$ and $f''(e) = 0$, or $f'(e) = 0$ and $f''(e) = 1$; for each $e \notin E_{f_1=\pm 1} = E_{f_2=\pm 1}$, $f'(e) + f''(e) = \pm 2$.

$\Leftarrow$: Assume that the graph $G$ admits a $(3, 3)$-flow even-disjoint-pair-cover $\{(D, f'), (D, f'')\}$. Then

\[ \{(D, f_2) = (D, f' + f''), (D, f_4) = (D, f' - f'')\} \]

is a $(3, 3)$-flow parity-pair-cover of $G$. \[\blacksquare\]

6. Cycle double covers

**Theorem 6.1.** Let $G$ be a graph. Then $G$ admits a $(4, 4)$-flow even-disjoint-pair-cover $\{(D, f_1), (D, f_2)\}$ if and only if $G$ has a 5-cycle double cover (which contains $E_{f_1=\text{odd}} \Delta E_{f_2=\text{odd}}$ as a member).

**Proof.** $\Rightarrow$: Assume that $\{(D, f'), (D, f'')\}$ is a $(4, 4)$-flow even-disjoint-pair-cover of $G$. Consider $A = E_{f'=\text{even}, f'' \neq 0}, B = E_{f''=\text{even}, f' \neq 0}$, and $C = E_{f'=\text{odd}} \cup E_{f''=\text{odd}}$, as a partition of $E(G)$. By Lemma 2.5, $\supp(f')$ has a 2-cycle cover, and so does $\supp(f'')$. The set of these four cycles covers each edge $e$ twice if $e \in A \cup B = E(G) \setminus C$ or $e \in E_{f'=\text{odd}} \cap E_{f''=\text{odd}}$, and once if $e \in E_{f'=\text{odd}} \Delta E_{f''=\text{odd}}$.

$\Leftarrow$: Assume that $\{C_1, \ldots, C_5\}$ is a 5-cycle double cover of $G$. Let $(D, f_i)$ be an integer 2-flow of $G$ with $\supp(f_i) = E(C_i)$ for each $i \in \{1, 2, 3, 4\}$. Let $C_i = C_1 \Delta C_i$ and $(D, f_{ij})$ be an integer 2-flow of $G$ with $\supp(f_{ij}) = E(C_{ij})$ for each $\{i, j\} \subset \{1, 2, 3, 4\}$. Then $\{(D, f_{12} + 2f_2), (D, f_{34} + 2f_4)\}$ is a $(4, 4)$-flow even-disjoint-pair-cover with

\[ E_{f_{12}+2f_2=\text{odd}} \Delta E_{f_{34}+2f_4=\text{odd}} = E(C_5). \]

**Theorem 6.2.** If $G$ has an orientable 5-cycle double cover, then $G$ has a $(3, 3)$-flow even-disjoint-pair-cover.

**Proof.** Assume that $\{C_1, \ldots, C_5\}$ is an orientable 5-cycle double cover of $G$ with each $C_i$ associated with an Eulerian orientation $D_i$, $\mu = 1, \ldots, 5$, such that for each edge $e$, both orientations are used in the corresponding $C_i$’s. Let $(D_{\mu}, f_{\mu})$ be a non-negative integer 2-flow of $G$ with support $C_{\mu}$. Let $D$ be an arbitrary orientation of $G$, and let $(D, f_{\mu})$ be an integer 2-flow of $G$ with $f_{\mu}(e) = f_{\mu}(e)$ if the edge $e$ has the same orientation of $D$ and $D_{\mu}$, and $f_{\mu}(e) = -f_{\mu}(e)$ if the edge $e$ has the opposite orientations of $D$ and $D_{\mu}$. Thus, $\{(D, f_1 - f_2), (D, f_3 - f_4)\}$ is a $(3, 3)$-flow even-disjoint-pair-cover of $G$. \[\blacksquare\]

Note that the existence of 5-cycle double covers and orientable 5-cycle double covers are conjectured for all bridgeless graphs.

**Conjecture 6.3** (Preissmann [9] and Celmins [2]). Every bridgeless graph has a 5-cycle double cover.

**Conjecture 6.4** (Jaeger [8]). Every bridgeless graph has an orientable 5-cycle double cover.

The following proposition was observed by one of the referees of this paper without applying the 4-color theorem.

**Proposition 6.5.** Every planar graph has a $(3, 3)$-flow parity-pair-cover.

**Proof.** Since every planar bridgeless graph has a proper 5-face-coloring, and thus has an orientable 5-cycle double cover (each cycle is defined by the face boundaries of a color class). Now apply Theorem 6.2 and subsequently Theorem 5.1. \[\blacksquare\]
7. Inclusion relations

By the theorems above, we have the following relations (as a poset):

\[ G = G_{(3,4)}\text{-parity-pair} = G_{F6} \]
\[ G_{F5} \subseteq G_{5\text{CDC}} \]
\[ G_{(3,3)}\text{-parity-pair} = G_{(3,3)}\text{-even-disjoint-pair} \]
\[ G_{O5\text{CDC}} \]
\[ G_{F4} = G_{(3,3)}\text{-Strong-parity-pair} \]
\[ G_{F3} \]

Some explanations about the poset:

- \( G \) is the family of all bridgeless graphs;
- \( G_{Fk} \) is the family of graphs admitting nowhere-zero \( k \)-flows;
- \( G_{(h,k)}\text{-parity-pair} \) is the family of graphs admitting \( (h,k) \)-flow parity-pair-covers;
- \( G_{(h,k)}\text{-even-disjoint-pair} \) is the family of graphs admitting \( (h,k) \)-flow even-disjoint-pair-covers;
- \( G_{(h,k)}\text{-strong-parity-pair} \) is the family of graphs admitting \( (h,k) \)-flow strong parity-pair-covers;
- \( G_{5\text{CDC}} \) is the family of graphs admitting 5-cycle double covers;
- \( G_{O5\text{CDC}} \) is the family of graphs admitting orientable 5-cycle double covers;
- The symbol "\( \subseteq \)" indicates that one family of graphs is a proper subset of another family;
- The symbol "\( \subset \)" indicates that one family of graphs is a subset of another family, and that is further conjectured that they are the same.

For example, by Theorems 5.1 and 6.1, \( G_{O5\text{CDC}} \subseteq G_{(3,3)}\text{-parity-pair} \) and it is conjectured that \( G_{O5\text{CDC}} = G_{(3,3)}\text{-parity-pair} \) (Conjecture 6.3). Note that \( G_{F4} \subset G_{O5\text{CDC}} \) since the Petersen graph is in \( G_{O5\text{CDC}} - G_{F4} \).

The following figures describe a pair of 3-flows in \( P_{10} \) which form a \((3,3)\)-flow parity-pair-cover.

*The Petersen graph \( P_{10} \)*
A 3-flow \((D, f_1)\) of \(P_{10} - f:\)

\[ f_1(e) = \pm 1 \text{ for each edge } e \text{ belonging to the dominating circuit,} \]

\[ f_1(e) = \pm 2 \text{ for all other edges.} \]

\[ f_2(e) = \pm 1 \text{ for each edge } e \text{ belonging to the dominating circuit,} \]

\[ f_2(e) = \pm 2 \text{ for all other edges.} \]

8. Open problems

**Conjecture 8.1.** Every bridgeless cubic graph \(G\) admits a \((3, 3)\)-flow parity-pair-cover.

With Tutte’s 5-flow Conjecture, the following is a weak version of Conjecture 8.1.

**Conjecture 8.2.** If \(G\) admits a nowhere-zero 5-flow, then \(G\) admits a \((3, 3)\)-flow parity-pair-cover.

By Theorems 4.2 and 4.3, the \((3, 3)\)-flow parity-pair-cover property lies between the properties of 4-flow and 5-flow.

**Problem 8.3.** Can we find a rational number \(r : 4 \leq r \leq 5\) such that the \((3, 3)\)-flow parity-pair-cover property is equivalent to the circular \(r\)-flow property?

The following conjecture is equivalent to the 5-cycle double cover conjecture (because of Theorem 6.1).

**Conjecture 8.4.** Every graph \(G\) admits an \((4, 4)\)-flow even-disjoint-pair-cover \(\{(D, f_1), (D, f_2)\}\).

Acknowledgments

Second author was supported in part by the National Security Agency under Grant H98230-05-1-0080 and by a WV RCG grant.

References


