HOMOGENIZATION FOR A CLASS OF INTEGRAL FUNCTIONALS IN SPACES OF
PROBABILITY MEASURES

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Abstract. We study the homogenization of a class of actions with an underlying Lagrangian \( L \) defined on the set of absolutely continuous paths in the Wasserstein space \( \mathcal{P}_p(\mathbb{R}^d) \). We introduce an appropriate topology on this set and obtain the existence of a \( \Gamma \)-limit of the rescaled Lagrangians. Our main goal is to provide a representation formula for these \( \Gamma \)-limits in terms of effective Lagrangians. Not only does this allow us to study “convexity properties” of the effective Lagrangian, but also the differentiability properties of its Legendre transform restricted to constant functions. In the case \( d > 1 \) we obtain partial results in terms of an effective Lagrangian defined on \( L^p((0,1)^d;\mathbb{R}^d) \). Our study provides a way of computing the limit of a family of metrics on the Wasserstein space. The results of this paper can also be applied to study the homogenization of variational solutions of the one-dimensional Vlasov-Poisson system, as well as the asymptotic behavior of calibrated curves. Whereas our study in the one-dimensional case covers a large class of Lagrangians, that of the higher dimensional case is concerned with special Lagrangians such as the ones obtained by regularizing the potential energy of the \( d \)-dimensional Vlasov-Poisson system.

1. Introduction

Let us consider a continuous path defined on the set of probability measures on the phase space \( \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \): \( t \in [0,T] \rightarrow f_t \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \) and consider rescaled actions defined on such paths

\[
A^\epsilon(f) := \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} v^2_t f_t(dx, dv) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(\frac{x-y}{\epsilon}) f_t(dx) f_t(dy),
\]

whose critical points satisfy a nonlinear Vlasov system. When \( f_t \) is the push forward of a probability density \( \rho_t \) defined on the physical space \( \mathbb{R}^d \) by a map of the form \( \text{id} \times v_t \) defined on \( \mathbb{R}^d \times \mathbb{R}^d \) (where \( \text{id} \) is the identity map), the above actions are reduced to

\[
(1.1) \quad \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} |v_t(x)|^2 \rho_t(dx) - \int_0^T \int_{\mathbb{R}^d} W(\frac{x-y}{\epsilon}) \rho_t(dx) \rho_t(dy).
\]

In this manuscript we study the \( \Gamma \)-limit of a class of functionals that includes those appearing in (1.1) and their link with the homogenization of systems of PDE’s of the nonlinear Vlasov type. We endow \( AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d)) \), the set of \( p \)-absolutely continuous paths on \( \mathcal{P}_p(\mathbb{R}^d) \) \( [2] \), with a topology \( \tau_w \) for which the sublevel sets of our actions are pre-compact (cf. section 3). More precisely, \( \tau_w \) is defined as follows: we say that \( \{\sigma_n\}_n \subset AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d)) \) satisfying the condition

\[
\sup_{n \in \mathbb{N}} \int_0^T \left( W_p(\rho_n, \delta_0) + \|\rho_n\|_{L^p}^p \right) dt < \infty
\]

\( \tau_w \)-converges to \( \sigma \) in \( AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d)) \) if

\[
\lim_{n \to \infty} \int_0^T W_1(\sigma_n, \sigma) dt = 0.
\]
Here, \( W^q \) is the \( L^q \)-Wasserstein distance for \( 1 \leq q < \infty \) and \( |\sigma'| \) is the metric derivative of \( \sigma \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d)) \). For such paths \( \sigma \) and the associated (unique) velocity \( v \) of minimal \( L^p(\sigma) \)-norm \([2]\), we define

\[
F^c(\sigma) = \int_0^T \mathcal{L}(D^c \sigma_t, v_t \circ D^c)dt,
\]

where \( D^c \) is the map of \( \mathbb{R}^d \) onto itself defined by \( D^c x = cx \). The notation \( L^p \) replaces \( L^p \) when dealing with \( \mathbb{R}^d \)-valued \( p \)-integrable functions. If \( d = 1 \), we stick with the classical notation \( L^p \). We have denoted the push forward operator by \( \# : \mathcal{P}_p(\mathbb{R}^d) \to \mathcal{P}_p(\mathbb{R}^d) \). The topology \( \tau_w \) seems to be too weak to directly provide information on the \( \Gamma(\tau_w) \)-limit of the functionals \( F^c \). Our strategy is to introduce a stronger topology \( \tau \) on a subset of \( AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d)) \), hoping to be able to extract enough information from the \( \Gamma(\tau) \)-limit of the functionals \( F^c \) in order to draw some conclusions on the \( \Gamma(\tau_w) \)-limit. The strategy completely paid off when \( d = 1 \), and produced partial results in the multi-dimensional case. The \( \tau \)-topology we consider is inspired from the well-known isometry between the convex cone of nondecreasing functions in \( L^p(0,1) \) and the Wasserstein space \( \mathcal{P}_p(\mathbb{R}) \). Denote \( X := (0,1)^d \). We consider the set of pairs \((\sigma,v)\) such that \( \sigma \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d)) \), \( \sigma \) is its minimal norm velocity and there exists \( \phi \in AC^p(0,T;\mathcal{L}(X)) \) satisfying the following properties:

\[
\dot{\phi}_t = v_t \circ \phi_t \text{ for almost every } t \in (0,T) \text{ and } \phi_t \text{ pushes } \nu_0 \text{ (the Lebesgue measure on the unit cube) forward to } \sigma_t \text{ for all } t \in [0,T].
\]

When these conditions are satisfied, we say that \( \phi \) is a flow associated to \((\sigma,v)\). We denote by \( \overline{S}_p(\mathbb{R}^d) \) the set of such special curves \( \sigma \). When \( d = 1 \), \( AC^p(0,T;\mathcal{P}_p(\mathbb{R})) \) and \( \overline{S}_p(\mathbb{R}) \) coincide \([13]\).

Let \( \{\sigma^n\}_{n \in \mathbb{N}} \subset \overline{S}_p(\mathbb{R}^d) \) and suppose \( \nu^n \) is the velocity of minimal norm for \( \sigma^n \). We say that \( \{\sigma^n\}_{n} \) \( \tau \)-converges to \( \sigma \in \overline{S}_p(\mathbb{R}^d) \) if there exists a sequence \( \{\phi\} \cup \{\phi^n\}_{n} \subset AC^p(0,T;\mathcal{L}(X)) \) such that \( \phi^n \) (and \( \phi \)) is a flow associated to \( \sigma^n \) (and \( \sigma \), respectively) and

\[
\|\phi^n - \phi\|_{L^p(0,T \times X)} \to 0 \text{ and } \{\phi^n\}_{n}, \{\phi^n\}_{n} \text{ are bounded in } L^p((0,T) \times X)
\]

for any \( Y \subset X \). We only consider Lagrangians \( \mathcal{L} \) for which there exist real valued functionals \( L \) defined on \( L^p(X) \times L^p(X) \) such that \( \mathcal{L}(\mu,\nu) = L(M,\nu \circ M) \) if \( \mu \in \mathcal{P}_p(\mathbb{R}^d), \nu \in L^p(\mu) \) and \( M \in L^p(X) \) is a map which pushes \( \nu_0 \) to \( \mu \).

In the first part of this work we study the \( \Gamma(\tau) \)-limit of \( F^c \) or, equivalently, the \( \Gamma(\tau_w) \)-limit when \( d = 1 \), without imposing any convexity assumption on \( L(M,\cdot) \). Our study on \( S := AC^p(0,T;\mathcal{P}_p(\mathbb{R})) \) is greatly facilitated by the well-known isometry mentioned above. We considerably exploit the established fact that if \( \sigma \in S \) and \( v \) is the velocity of minimal norm for \( \sigma \), then \( \phi_t = v_t \circ \phi_t \) for almost every \( t \in (0,T) \) (cf. \([13]\)). Here, \( \phi_t \) is the monotone nondecreasing map that pushes \( \nu_0 \) forward to \( \sigma_t \). Under some mild continuity assumptions on \( L \), we establish a representation formula for \( F^c \), the \( \Gamma(\tau_w) \)-limit of \( F^c \), and for \( F \), the \( \Gamma(\tau) \)-limit of \( F^c \), in terms of the effective Lagrangian \( L \) of \( L \):

\[
F(\sigma) = \int_0^T \tilde{L}(\nu_t \circ \phi_t)dt = F_w(\sigma).
\]

We extend some of the results obtained in the one-dimensional setting to the higher dimensional one, where the Lagrangians \( \mathcal{L}^c \) are defined on the “tangent bundle” of the Wasserstein space \( \mathcal{P}_p(\mathbb{R}^d) \). To simplify the arguments in the second part of this manuscript, we restrict ourselves to Lagrangians assuming the special form \( \mathcal{L}(\mu,\xi) = ||\xi||_p/p + W(\mu) \), where \( W \) is continuous, real valued periodic function, in a sense to be specified.

For \( \mu \in \mathcal{P}_p(\mathbb{R}^d) \) and \( \xi \in L^p(\mu) \) it is readily seen that the expression

\[
\tilde{L}(\mu,\xi) = \lim_{T \to \infty} \frac{C_{0,T}(\mu_0, (T\xi)_{\#}\mu)}{T}
\]

is independent of \( \mu_0 \in \mathcal{P}_p(\mathbb{R}^d) \). Here, \( C_{0,T}(\mu_0, (T\xi)_{\#}\mu) \) is the infimum of \( \int_0^T \mathcal{L}(\sigma_t, v_t)dt \) over the set of \((\sigma,v)\) such that \( \sigma \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d)) \), \( v \) is a velocity for \( \sigma \) and \( \sigma_0 = \mu_0, \sigma_T = (T\xi)_{\#}\mu \). We refer to \( \tilde{L} \) as the effective Lagrangian associated to \( \mathcal{L} \). Note that no topology need be mentioned in connection with the definition of \( \tilde{L} \) and so, this functional is intrinsic.
For the strong topology $\tau$ chosen in subsection 5.3, existence of the $\Gamma(\tau)$-limit $F$ of $F^\epsilon$ is obtained by standard arguments. We show that

$$F(\sigma) = \int_0^T \tilde{L}(\sigma_t, v_t)dt = \int_0^T \tilde{L}(v_t \circ \phi_t)dt$$

for a convex functional $\tilde{L} : L^p(X) \to \mathbb{R}$, which is precisely the effective Lagrangian of $L$. Here $v$ is the velocity of minimal norm for $\sigma$ and $\phi$ is any flow associated to $(\sigma, v)$. What is obvious is that $F_w \leq F$ on $S_p(\mathbb{R}^d)$.

The harder and more interesting question is to obtain a representation formula for $F_w$. In higher dimension, only for a dense subset of $P_p(\mathbb{R}^d)$ were we able to establish the representation formula

$$F_w(\sigma) = F(\sigma) = \int_0^T \tilde{L}(v_t \circ \phi_t)dt.$$  

More precisely, we have proved that (1.6) holds if the path $\sigma$ is an averages of finitely many Dirac masses and is piecewise a geodesic in $P_p(\mathbb{R}^d)$. We also establish this representation formula for $\sigma$ of the form $\sigma_t = (\xi_\epsilon)\#\nu_0$ where $\xi \in L^p(X)$. It remains unclear if one can extend that result to paths of the form $\sigma_t = (\text{id} + t\xi)\#\nu_0$. This seems to be the obstruction to extending the representation formula to arbitrary elements of $AC^p(0,T; P_p(\mathbb{R}^d))$.

There is a vast literature in the finite dimensional setting, on the $\Gamma$-limit theory of functionals of the form

$$\int_0^T l(r(\epsilon, \dot{r})dt$$

chosen in subsection 5.3, existence of the $\Gamma(\epsilon)$-limit $F^\epsilon$ can be used to write a representation formula for $F^\epsilon$. This is the Lagrangian for the one-dimensional Vlasov-Poisson system. Let $\sigma^{\epsilon*}$ be a critical point of $F^\epsilon$ and let $v^{\epsilon*}$ be the velocity of minimal norm for $\sigma^{\epsilon*}$. The path $\sigma^{\epsilon*}$ induces a path $g^\epsilon$ on $P_p(\mathbb{R} \times \mathbb{R})$, which itself induces a path $f^\epsilon$ on $P_p(\epsilon \mathbb{T}^1 \times \mathbb{R})$ as follows:

$$g^\epsilon_t = (\text{id} \times v^\epsilon_t)\# \sigma^{\epsilon*}, \quad f^\epsilon_t = \sum_{k \in \mathbb{Z}} g^\epsilon_t(x + \epsilon k, v).$$

According to [16], $f^\epsilon$ is a solution of the Vlasov-Poisson system, $\epsilon$-periodic in the space variable:

$$\begin{cases}
\partial_t f^\epsilon_t + v\partial_x f^\epsilon_t = \frac{1}{\epsilon} \partial_x P^\epsilon_t \partial_v f^\epsilon_t \\
1 - \epsilon \partial_x P^\epsilon_t = \epsilon \rho^\epsilon_t \\
\rho^\epsilon_t = \int_{\mathbb{R}} f^\epsilon(\cdot, dv).
\end{cases}$$

For variational solutions of (1.7), using standard arguments of the $\Gamma$-limit theory we have been able to characterize the points of accumulation of $\{f^\epsilon\}_\epsilon$.

Now let us consider $L$ so that, given $\mu_0, \mu_1 \in P_2(\mathbb{R})$, the infimum of $F^\epsilon(\sigma)$ over the set of paths $\sigma$ with endpoints $\mu_0$ and $\mu_1$ is a number $(\text{dist}^\epsilon(\mu_0, \mu_1))^2$ such that $\text{dist}^\epsilon$ is a metric. As $AC^2(0,T; P_2(\mathbb{R}))$ is a separable space, it is easy to obtain a subsequence $\{\text{dist}^\epsilon_n\}_n$ of these metrics which converges pointwise to a limit $\text{dist}$. The $\Gamma$-limit of $F^\epsilon$ can be used to write a representation formula for $\text{dist}$. We use these results to study homogenization of Hamilton-Jacobi equations on the Wasserstein space.

The class of Lagrangians covered by our study includes those of systems of two interacting species in motion on the real line:

$$L_4(\mu_1, \mu_2, \xi_1, \xi_2) = \frac{1}{4} ||\xi_1||_{\mu_1}^2 + \frac{1}{4} ||\xi_2||_{\mu_2}^2 - \int_{\mathbb{R}^2} W(x - y)\mu_1(dx)\mu_2(dy).$$
Here, $W$ is a periodic, bounded function defined on the real line. We also consider Lagrangians of systems of interacting one-species (resp. non-interacting particles):

$$L_2(\mu, \xi) = L_2(\mu, \mu, \xi, \xi), \quad \text{(resp. } L_1(\mu, \xi) = \|v\|^2/2 - \int_{\mathbb{R}} W d\mu).$$

Our strategy is to prove under suitable hypotheses that the action of $L'(M, N) := L(M/\epsilon, N)$, converges (in the $\Gamma$-convergence sense) on curves in $\mathcal{H}$ to the action of a homogenized Lagrangian $\bar{L}$. Most of the techniques of E [7] apply to our infinite-dimensional system as well. We can prove that $\bar{L}_1(N) = \int_X l_1 \circ N dx$, where $l_1$ is the Lagrangian defined over $\mathbb{R}^2$ by $l_1(x, v) = v^2/2 - W(x)$ [7], [17] whose effective Lagrangian $\bar{l}_1$ is computed “explicitly” in [17].

Two important consequences will emerge: we obtain the homogenization for the time-dependent Hamilton-Jacobi equations on $(0, \infty) \times \mathcal{P}(\mathbb{R})$ associated to $\mathcal{L}'$. Indeed, it is a remarkable fact that one is able to use a $\Gamma$-convergence result for the “extended” Lagrangians $L'$ to make inferences about $\mathcal{L}'$. This comes as a consequence of the fact that the optimal paths/characteristics in the variational solutions for the Hamilton-Jacobi equations considered here lie pointwise in the set of monotone nondecreasing functions in $L^p(0, 1)$ (this was observed in [13]) if the endpoints do so.

Secondly, we will obtain the convergence of action minimizing solutions for systems of the type

$$\begin{align*}
\partial_t \rho'^{\epsilon} + \partial_x (\rho' v'^{\epsilon}) &= 0, \\
\partial_t (\rho' v'^{\epsilon}) + \partial_x [\rho' (v'^{\epsilon})^2] &= -1/\epsilon \rho' A'_1,
\end{align*}$$

(1.8)

to minimizers of the action $\int_0^T \bar{L}_1(M) dt$, for

$$A'_1(x) = V_1\left(\frac{x}{\epsilon}\right) \quad \text{and} \quad A'_2(t, x) = \int V_1\left(\frac{x-y}{\epsilon}\right) \rho'(t, dy),$$

respectively.

2. Homogenization on $L^p(X)$

For any open set $\Omega \subset \mathbb{R}$ we denote by $\mathcal{H}_\Omega$ the set of $\phi \in W^{1, p}(\Omega; L^p(X))$ which admit extensions $\tilde{\phi} \in W^{1, p}_{loc}(\mathbb{R}; L^p(X))$, and we write $\mathcal{H}$ in place of $\mathcal{H}_{(0, T)}$ when there is no confusion.

Let $L$ be a Lagrangian on $L^p(X) \times L^p(X)$ satisfying the growth conditions

$$c\|N\|^p \leq L(M, N) \leq C(1 + \|N\|^p), \quad \text{for all } M, N \in L^p(X),$$

(2.1)

where $c$, $C$ are given positive constants. We set $L'(M, N) = L(M/\epsilon, N)$, and assume $L$ is 1-periodic in $M$, i.e.

$$L(M + Z, N) = L(M, N) \quad \text{for all } Z \in L^p_2(X), \quad M, N \in L^p(X),$$

where we adopt the notation

$$L^p_2(Q) = \{Z \in L^p(X) : Z(X) \subset Q^d\}.$$

(2.2)

Also, we will use the sets

$$L^p_2(Q) = \{Z : X \rightarrow \mathbb{R}^d \text{ Lebesgue measurable} : Z(Q) \text{ is a finite subset of } \mathbb{Z}^d\}$$

(2.3)

and

$$L^p_2(Q) = \{Q : X \rightarrow \mathbb{R}^d \text{ Lebesgue measurable} : Q(X) \text{ is a finite subset of } \mathbb{Q}^d\}.$$

(2.4)

Given an open set $\Omega \subset \mathbb{R}$, we identify $L^p(\Omega \times X)$ with $L^p(\Omega; L^p(X))$. We refer to $t \in \Omega$ as the time variable and often write $\phi_t$ in place of $\phi(\cdot, t)$. This should not be confused with the partial derivative $\partial_t \phi$. If $\phi \in \mathcal{H}_\Omega$ we identify $\phi$, the Fréchet derivative of $t \rightarrow \phi_t$ with the distributional derivative of $\phi$ with respect to the time variable.
2.1. **The effective Lagrangian in** $L^p(X)$. Given $M_a, M_b \in L^p(X)$, $\epsilon > 0$ and $a < b$ we set

$$C_{a,b}(M_a, M_b) = \inf_M \left\{ \int_a^b L\left( \frac{\phi}{\epsilon}, \phi_t \right) dt : \phi \in \mathcal{H}_{(a,b)}, \phi_a = M_a, \phi_b = M_b \right\}$$

and write $C_{a,b}$ in place of $C'_{a,b}$. Observe that if $a < c < b$ and $M_c \in L^p(X)$, then

$$C_{a,b}(M_a, M_b) \leq C_{a,c}(M_a, M_c) + C_{c,b}(M_c, M_b)$$

and so, by (2.1) whenever $M_1, M_2, N \in L^p(X)$ and $S > 2$ we have

$$C_{0,S}(M_a, M_1 + S N) \leq C_{0,1}(M_a, M_b) + C_{0,(S-2)}(M_b, M_2 + (S-2)N) + C(1 + \|2N + M_1 - M_2\|^p)$$

Hence,

$$\frac{C_{0,S}(M_a, M_1 + S N)}{S} \leq \frac{C_{0,S}(M_b, M_2 + S N)}{S}$$

Since $(M_a, M_1)$ and $(M_2, M_b)$ are arbitrary, we conclude that the two limits in (2.7) coincide and so,

$$\bar{L}(N) := \liminf_{S \to \infty} \frac{C_{0,S}(M_a, M_1 + S N)}{S}$$

is independent of $M_a, M_1 \in L^p(X)$. If $\epsilon, T > 0$ and $\phi \in \mathcal{H}$ then

$$\epsilon \int_0^T L(\phi_t, \dot{\phi}_t) ds = \int_0^T L\left( \frac{\phi_t}{\epsilon}, \dot{\phi}_t \right) dt,$$

where $\epsilon \phi_t := \phi_{\epsilon t}$. Hence, $\epsilon C_{0,T/\epsilon}(\phi_0/\epsilon, \phi_T/\epsilon) = C_{0,T}(\phi_0, \phi_T)$ and so, in light of (2.8),

$$\bar{L}(N) = \liminf_{\epsilon \to 0^+} \frac{C_{0,T}(\epsilon M, TN)}{T} = \liminf_{\epsilon \to 0^+} \frac{C_{0,T}(0, TN)}{T}.$$

**Remark 2.1.** Suppose $M \in L^p(X), Z \in L^p(X)$ and $T > 0$. Then $T \bar{L}(Z/T) \leq C_{0,T}(M, M + Z)$. In particular, if $Q \in L^p(X)$ and $T > 0$ are such that $TQ \in L^p(X)$, then $C_{0,T}(M, M + TQ) \geq T \bar{L}(Q)$.

**Proof:** Let $\phi \in \mathcal{H}$ be such that $\phi_0 = \phi_T = 0$. Extend $t \to \phi_t$ to obtain a periodic function on $\mathbb{R}$, we still denote by $\phi$. Note that $t \to L(M + t/TZ + \phi_t, Z/T + \dot{\phi}_t)$ is $T$ periodic equal to $M$ at $t = 0$ and equal to $M + jZ$ at $t = jT$. Thus,

$$\int_0^T L(M + t/TZ + \phi_t, Z/T + \dot{\phi}_t)dt \geq \int_0^T L(M + t/TZ + \phi_t, Z/T + \dot{\phi}_t)dt \geq \frac{C_{0,T}(M, M + jZ)}{jT}.$$ 

We let $j$ tend to $\infty$, use (2.8) and take into consideration that $\phi \in \mathcal{H}$ is arbitrary and satisfies $\phi_0 = \phi_T$ to conclude the proof of the remark. 

2.2. **Time and space discretization of elements of** $\mathcal{H}$. In this subsection we recall a lemma and corollary which are, in different forms, well-known. Suppose $\phi \in \mathcal{H}$. Fix an integer $m > 1$ and set $t_j = j/mT = th$ for $j = 0, \cdots, m$. For $s \in [t_i, t_{i+1}]$ we set

$$\phi^m_{s,i} := \left( 1 - \frac{s - t_i}{h} \right) \phi_{t_i} + \frac{s - t_i}{h} \phi_{t_{i+1}}.$$

**Lemma 2.2.** We have

$$\int_0^T |(\phi^m)'|^p(s)ds \leq \int_0^T |\phi'|^p(s)ds, \|\phi_t - \phi^m_t\|^p \leq 2h^p C_{\sigma} \text{ for all } t \in [0, T],$$

where $C_{\sigma}^p := \int_0^T |\phi'|^p(s)ds$. Hence, $(\phi^m_t)_N$ converges to $\phi_t$ in $L^p(X)$ for $t \in [0, T]$. If $|\phi'| \leq C_{\sigma}$ almost everywhere on $(0, T)$ then $|(\phi^m)'| \leq C_{\sigma}$ almost everywhere on $(0, T)$. 

Corollary 2.3. Let $\phi \in \mathcal{H}$ and let $\delta > 0$. Then there exist a partition of $X$ into finitely many parallel cubes $\{X_i\}_{i=1}^k$ of same size and $\psi \in \mathcal{H}$ such that $\|\phi_t - \psi_t\| \leq \delta$ for all $t \in [0, T]$, for $t$ fixed $\psi_t$ is constant on each cube, its range is contained in $Q^d$ and

$$\int_0^T \|\dot{\psi}_t\|^p dt \leq \int_0^T \|\dot{\phi}_t\|^p dt + \delta$$

and $\psi$ can be chosen so that $|\psi'| \leq C_\tau + 1$ and the number of cubes can be chosen to be of the form $m^d$.

2.3. Topologies on $\mathcal{H}_\Omega$. Let $\Omega \subset \mathbb{R}$ be an open set. We endow $\mathcal{H}_\Omega$ with the topology $\tau_\Omega$ given by

$$\phi^n \xrightarrow{\tau_\Omega} \phi \iff \|\phi^n - \phi\|_{Lp(\Omega \times Y)} \to 0$$

and

$$\{\phi^n\}_n \text{ are bounded in } L^p(\Omega \times X)$$

for any $Y \subset X$. Observe that $\{\phi^n\}_n \subset \mathcal{H}_\Omega$ converges to $\phi$ in $\mathcal{H}_\Omega$ if and only if the following three conditions are satisfied:

$$\begin{align*}
(\text{ii}) & \quad \{\phi^n\}_n \text{ converges to } \phi \text{ in } L^p(\Omega \times Y), \text{ for all } Y \subset X; \\
(\text{ii}) & \quad \{\phi^n\}_n \text{ converges weakly to } \phi \text{ in } L^p(\Omega \times X); \\
(\text{ii}) & \quad \{\phi^n\}_n \text{ converges weakly to } \phi \text{ in } L^p(\Omega \times X).
\end{align*}$$

If (2.13–2.15) hold then

$$\int_\Omega (\|\phi^n\|^p + \|\dot{\phi}^n\|^p) dt < \infty.$$
Sketch of proof: Let $\mathcal{D}$ be the algebra generated by the open bounded intervals in $\mathbb{R}$ whose endpoints are rational number. Fix for a moment an integer $e > 0$ and $\Omega \in \mathcal{D}$. Let $\mathcal{H}_e^\infty$ be the set of $M \in \mathcal{H}_e$ such that $\int_\Omega ||M||^p dt \leq e$ and denote by $\tau_e^\infty$ the restriction of the topology $\tau_e$ to $\mathcal{H}_e^\infty$. Note that $(\mathcal{H}_e^\infty, \tau_e^\infty)$ is a topological space which has a countable basis. Hence, there exists a sequence $\{\epsilon_k\}_{k \in \mathbb{N}}$ converging to zero as $k$ tends to $\infty$ such that

$$\Gamma(\tau_e) \lim_{k \to \infty} F^{\epsilon_k}(\phi, \Omega) \text{ exists,}$$

for $\phi \in \mathcal{H}_e^\infty$. The growth condition (2.1) ensures that

$$\Gamma(\tau) \lim_{k \to \infty} F^{\epsilon_k}(\phi, \Omega) \text{ exists,}$$

for $\phi \in \mathcal{H}_e^\infty$. Since $e \in \mathbb{N}$ is arbitrary and $\mathcal{D}$ is countable, using a diagonalization argument we obtain the following: there exists a sequence $\{\epsilon_n\}_{n \in \mathbb{N}}$ converging to zero as $n$ tends to $\infty$ such that there exists

$$\Gamma(\tau) \lim_{n \to \infty} F^{\epsilon_n}(\cdot, \Omega)$$

for any $\Omega \in \mathcal{D}$. Let $\mathcal{O}_b$ be the set of bounded open sets $\Omega \subset \mathbb{R}$. Denote by $F^-(-, \Omega)$ (resp. $F^+(\cdot, \Omega)$) the lower $\Gamma$–limit (resp. upper $\Gamma$–limit) of $\{F^{\epsilon_n}(\cdot, \Omega)\}_{n \in \mathbb{N}}$ for any open set $\Omega$. According to the existence of the $\Gamma$–limit above, we have that $F^-(\phi, A) = F^+(\phi, A)$ for all $\phi \in \mathcal{H}$ and any $A \in \mathcal{D}$. Thus, for $\phi \in W^{1,p}_{loc}(\mathbb{R}; L^p(X))$ we can define $F(\phi, \cdot)$ on $\mathcal{D}_b$ by setting

$$F(\phi, \Omega) = A_{\in \mathcal{D}} \sup_{A \in \mathcal{D}, A \subset \Omega} F^-(\phi, A) \sup_{A \in \mathcal{D}, A \subset \Omega} F^+(\phi, A).$$

We would like to show that $F(\cdot, \Omega)$ is the $\Gamma$–limit of $F^{\epsilon_n}(\cdot, \Omega)$ for all $\Omega \in \mathcal{O}_b$. For fixed $\phi$, one can easily show that $F^-(\phi, \cdot)$ is a superadditive set function. To show that $F^+(\phi, \cdot)$ is superadditive requires more work, but the same argument as in [7] applies. We claim that $F^-(-, \Omega) = F^+(\phi, \Omega) = F(\phi, \Omega)$ for any $\Omega \in \mathcal{O}_b$. For any $A \in \mathcal{D}$ with $A \subset \Omega$ we have

$$F^+(\phi, \Omega) \leq F^+(\phi, A) + F^+(\phi, \Omega \setminus A).$$

The growth properties (upper bound) on $L$ ensure that for any $\delta > 0$ one can choose $A$ appropriately such that $F^+(\phi, \Omega \setminus A) < \delta$. Thus, since $F^-$ is superadditive, we get that

$$F^+(\phi, A) = F^-(\phi, A) \leq F^-(\phi, \Omega) \leq F^+(\phi, \Omega) \leq F^+(\phi, A) + \delta = F(\phi, A) + \delta.$$

Since $\delta$ is arbitrary, the claim is proved. For one thing, the equality of $F^-$ and $F^+$ shows that $F$ is the $\Gamma$–limit of $\{F^{\epsilon_n}\}_n$. Also, we get that $F(\phi, \cdot)$ is an increasing, inner regular, finitely additive set function. Thus, it can be extended to a Borel measure on $\mathbb{R}$. The growth estimate (2.19) follows immediately from (2.1). \qed

We first ask the question of what $F$ is when $\phi$ is affine in time. For this, we have the following proposition whose main conclusion is that $F(\phi, \cdot)$ is a dilation of the Lebesgue measure.

**Proposition 2.6.** If $\phi(t, \cdot) = t \xi$ for some $\xi \in L^p(X)$, then $F(\phi, \Omega) = \Phi(\xi)|\Omega|$ for some lower semicontinuous functional $\Phi : L^p(X) \to \mathbb{R}$ satisfying

$$c ||\xi||^p \leq \Phi(\xi) \leq \mathcal{C}(1 + ||\xi||^p).$$

**Proof:** It is enough to show that if $a \in \mathbb{R}$, $f \in L^p(X)$, then the first equality below

$$F(f, \Omega + a) = F(\phi, \Omega) = F(\phi + f, \Omega)$$

holds for all $\Omega \in \mathcal{G}_b$ (open, bounded sets in $\mathbb{R}$ with negligible boundaries). The second equality will be needed towards the end of the section. Let $\phi_n \in W^{1,p}_{loc}(\mathbb{R}; L^p(X))$ such that $\phi_n \to \phi$ in $L^p(X)$ and $F(\phi, \Omega) = \lim F(\phi_n, \Omega)$. Now let $\varphi_n \in C_c(\mathbb{R}; \mathbb{R}^d)$ such that $\|\varphi_n - (\alpha/\epsilon_n)\xi\| < 1$ for all $n$. Take $Z_n = [\varphi_n]$ (vector consisting of the integer parts of all entries) and note that $Z_n \in L^a_{\mathbb{Z}}(X)$. We have that $\|Z_n - \varphi_n\| < 1$ implies $\|Z_n - (\alpha/\epsilon_n)\xi\| < 2\epsilon_n$. Put

$$N_n(s, x) = \phi_n(s - \alpha, x) + \epsilon_n Z_n(x), n \geq 1.$$
Clearly, \( N_n \rightarrow^\tau_{\Omega+\alpha} (s-\alpha)\xi + \alpha \xi \) in \( \mathcal{H}_{\Omega+\alpha} \), and

\[
F(\phi, \Omega) = \lim_{n \to \infty} \int_{\Omega} L \left( \frac{\phi_n}{\epsilon_n}, \phi_n \right) \, dt = \liminf_{n \to \infty} \int_{\Omega+\alpha} L \left( \frac{N_n}{\epsilon_n}, \hat{N}_n \right) \, dt \geq F(\phi, \Omega + \alpha).
\]

Of course, by symmetry, we obtain equality. As for the second inequality in (2.21), let \( P_n(t, x) = \phi_n(t, x) + \epsilon_n K_n \), where \( K_n \in \mathbf{L}_2(X) \) is such that \( \epsilon_n K_n \to f \) in \( \mathbf{L}^p(X) \). Thus, \( P_n \to^\tau f + M + f \) and

\[
F(\phi, \Omega) = \lim_{n \to \infty} \int_{\Omega} L \left( \frac{\phi_n}{\epsilon_n}, \phi_n \right) \, dt = \liminf_{n \to \infty} \int_{\Omega+\alpha} L \left( \frac{K_n}{\epsilon_n}, \hat{K}_n \right) \, dt \geq F(\phi + f, \Omega).
\]

Since here we have not used that \( \phi \) is affine, we get the reverse inequality by symmetry again, so we have equality. The growth conditions on \( \Phi \) are derived from (2.19) and its lower semicontinuity comes from the \( \Gamma \)-limit property of \( F \). \( \square \)

We plan to identify \( \Phi \) with \( \bar{L} \). To be able to link \( F(\cdot, \Omega) \) to \( \bar{L} \) we need to make additional conditions on \( L \). We assume that there exists \( \Lambda > 0 \) such that

\[
L(M, N_1) - L(M, N_2) \leq \Lambda \int_{\{N_1 \neq N_2\}} |N_1|^p \, dx
\]

for all \( M, N_1, N_2 \in \mathbf{L}^p(X) \). Also, \( L \) satisfies, for some continuous, monotone nondecreasing \( \omega : \mathbb{R} \to \mathbb{R} \) such that \( \omega(0) = 0 \),

\[
|L(M_1, N) - L(M_2, N)| \leq \omega \left( L^d \left( \{ M_1 \neq M_2 \} \right) \right)
\]

for all \( M_1, M_2, N \in \mathbf{L}^p(X) \).

The proof of the following proposition is an easy consequence of Lemmas 2.8, 2.10 and 2.12.

**Proposition 2.7.** Under (2.22) and (2.23) we have \( \Phi = \bar{L} \) on \( \mathbf{L}^p(X) \).

We begin by proving the following result.

**Lemma 2.8.** Then one has \( \bar{L}(\xi) \leq \Phi(\xi) \) for all \( \xi \in \mathbf{L}^p(X) \).

**Proof:** Take \( \Omega = (0, 1) \) and \( \phi_n \in \mathcal{H}_\Omega \) such that

\[
\phi_n \to^\tau 0 \quad \text{and} \quad F^{x_n}(t \xi + \phi_n, \Omega) \to \Phi(\xi).
\]

Assume for the moment \( \phi_n \in \mathcal{H}^0_\Omega := W^{1,p}_0(\Omega; \mathbf{L}^p(X)) \) and take \( N_n(t, x) = \phi_n(\epsilon_n t, x) / \epsilon_n, \Omega_n = \Omega / \epsilon_n \). Now,

\[
\Phi(\xi) = \lim_{n \to \infty} \int_{\Omega} L \left( \frac{t \xi + \phi_n}{\epsilon_n}, \xi + \phi_n \right) \, dt = \liminf_{n \to \infty} \epsilon_n \int_{\Omega_n} L(s \xi + N_n, \xi + \hat{N}_n) \, ds
\]

\[
\geq \liminf_{n \to \infty} \epsilon_n C_1 / \epsilon_n (0, \xi / \epsilon_n) \geq \bar{L}(\xi),
\]

where we have used (2.10). In fact, we have also proved that \( \liminf_{n \to \infty} \int_{\Omega} L \left( \frac{U_n}{\epsilon_n}, \hat{U}_n \right) \, dt \geq \bar{L}(\xi) \) for any sequence \( U_n \to^\tau t \xi \) such that \( U_n - t \xi \in \mathcal{H}_\Omega(\Omega) \). To eliminate this restriction (we need this inequality for sequences in \( \mathcal{H}_\Omega \) only), we shall use a technique due to De Giorgi [18]. In our setting, due to the more delicate nature of the topology \( \tau \) (strong \( L^p \) convergence is only local in space), a more subtle argument is needed. Let \( \{ U_n \} \subset \mathcal{H}_\Omega \) converge to \( t \xi \) in the \( \tau \) topology, and fix \( \delta > 0 \) sufficiently small. Let

\[
S_0 = (\delta, 1 - \delta) \times (\delta, 1 - \delta)^d \subset \cdots \subset S_N \subset S = (0, 1) \times (0, 1)^d
\]

be open, equidistant squares. Next we take

\[
a_i : [0, 1] \to [0, 1], \quad b_i : [0, 1]^d \to [0, 1]
\]

smooth functions such that

\[
a_i = b_i = 1 \text{ on } S_{i-1}, \quad a_i = b_i = 0 \text{ on } S \setminus S_i \quad \text{and} \quad |a_i(t)| \leq (N + 1) / \delta
\]
for all \( i = 1, \ldots, N \). Then consider
\[
U^i_n(t, x) = t\xi(x) + a_i(t)b_i(x)[U_n(t, x) - t\xi(x)]
\]
so that \( U^i_n - t\xi \in \mathcal{H}_0(\Omega) \). Note that
\[
F^{e_n}(U^i_n, \Omega) = F^{e_n}(U_n, \Omega) + \int_0^1 \left[ L\left( \frac{U_n}{e_n}, \dot{U}_n \right) - L\left( \frac{U_n}{e_n}, \dot{U}_n \right) \right] dt + \int_0^1 \left[ L\left( \frac{U_n}{e_n}, \dot{U}_n \right) - L\left( \frac{U_n}{e_n}, \dot{U}_n \right) \right] dt =: F^{e_n}(U_n, \Omega) + I^*_1 + I^*_2.
\]
Write \( S_i = I_i \times Q_i \) where \( I_i \subset (0, 1)^d \). Since \( U^i_n(t, \cdot) = U_n(t, \cdot) \) on \( Q_{i-1} \) for all \( t \in I_{i-1} \), we use (2.23) to infer that \(|I^*_2| \leq C\delta + \omega(C\delta)\). By (2.22)
\[
I^*_1 \leq \Lambda \int \int_{S \setminus S_{i-1}} |\dot{U}_n|^p dxdt.
\]
On \( S \setminus S_i \) we have \( U^i_n = t\xi \), so
\[
\int \int_{S \setminus S_i} |\dot{U}_n|^p dxdt \leq \int \int_{S \setminus S_i} |\xi(x)|^p dxd + \int \int_{S \setminus S_{i-1}} |\dot{U}_n|^p dxd + \int \int_{S \setminus S_{i-1}} |\dot{U}_n|^p dxd.
\]
Collecting all the above we get
\[
(F^{e_n}(U^i_n, \Omega) \leq F^{e_n}(U_n, \Omega) + \Lambda \int \int_{S \setminus S_{i-1}} |\dot{U}_n|^p dxd + C\delta + \omega(C\delta).
\]
Since
\[
|\dot{U}_n|^p = |\xi + a_i b_i(U_n - t\xi) + a_i b_i(\dot{U}_n - \xi)|^p \leq 3^p - 1 \left[ |\xi|^p + \left( \frac{N + 1}{\delta} \right)^p |U_n - t\xi|^p + |\dot{U}_n - \xi|^p \right],
\]
we infer
\[
F^{e_n}(U^i_n, \Omega) \leq F^{e_n}(U_n, \Omega) + 0(\delta) + 3^p - 1 \Lambda \left( \frac{N + 1}{\delta} \right)^p \int \int_{S \setminus S_{i-1}} |U_n - t\xi|^p dxd + C\delta + \omega(C\delta).
\]
	Since \( U_n \) converges to \( t\xi \) locally in \( L^p(S) \), upon passing to liminf as \( n \to \infty \), we obtain
\[
\liminf_{n \to \infty} F^{e_n}(U^i_n, \Omega) \leq \liminf_{n \to \infty} F^{e_n}(U_n, \Omega) + 3^p - 1 \Lambda \liminf_{n \to \infty} \int \int_{S \setminus S_{i-1}} |\dot{U}_n - \xi|^p dxd + C\delta + 0(\delta).
\]
Since \( \{U^i_n - t\xi\} \subset \mathcal{H}_0(\Omega) \), we know the left hand side is at least \( L(\xi) \). Thus, if we sum up from \( i = 1 \) to \( N \) and divide by \( N \), we find
\[
\frac{L(\xi)}{N} \leq \liminf_{n \to \infty} F^{e_n}(U_n, \Omega) + \frac{3^p - 1}{N} \Lambda \liminf_{n \to \infty} \int \int \int |\dot{U}_n - \xi|^p dxd + C\delta + 0(\delta).
\]
Due to \( U_n \to t\xi \), the integral is uniformly bounded in \( n \), so, if we first let \( N \to \infty \) and then \( \delta \to 0 \), we discover
\[
\bar{L}(\xi) \leq \liminf_{n \to \infty} F^{e_n}(U_n, \Omega).
\]
Thus, the proof is concluded. \( \square \)

If \( R \in L^q(X) \), then the map \( t \to L(tR, R) \) is periodic having as principal period the least common multiple of all the denominators of all (rational) entries of all vectors in the range of \( R \), say, \( m \). We next study \( L(R) \) for \( R \in L^q(X) \).
Lemma 2.9. One has
\[ \bar{L}(R) = \lim_{k \to \infty} \frac{C_{km}(0, kmR)}{km}. \]

We skip the proof of this lemma, as it needs no modification from the argument in [7]. On the basis of this result, we shall be able to obtain the opposite inequality, \( \bar{L} \geq \Phi \), at first on \( L_\mathbb{Q}(X) \).

Lemma 2.10. One has \( \bar{L}(R) \geq \Phi(R) \) for all \( R \in L_\mathbb{Q}(X) \).

Proof: Take \( \delta > 0 \). According to Lemma 2.9, we can find a positive integer \( k \) and \( \phi \in \mathcal{H}_0(0, km) \) such that
\[ \left| \bar{L}(R) - \int_0^{km} L(sR + \phi, R + \dot{\phi}) ds \right| < \delta. \]

Set \( \Omega = (0, 1) \) and define \( \tilde{\phi} \) as the periodic extension of \( \phi \) on \( \mathbb{R} \) and put \( M_n(t, x) := tR(x) + \epsilon_n \tilde{\phi}(t/\epsilon_n, x) \). Then \( M_n \to tR \), so
\[ \liminf_{n \to \infty} \int_0^1 L\left( \frac{M_n}{\epsilon_n}, M_n \right) dt \geq F(tR, \Omega) = \Phi(R). \]

We have
\[ \int_0^1 L\left( \frac{M_n}{\epsilon_n}, M_n \right) dt = \int_0^{T_n} L(tR + \tilde{\phi}, R + \dot{\tilde{\phi}}) dt, \]
where \( T_n := 1/\epsilon_n \). The map
\[ t \to L(tR + \tilde{\phi}, R + \dot{\tilde{\phi}}) \]
is periodic of period \( km \). Thus, the last averaged integral from above becomes
\[ \frac{T_n}{km} \int_0^{km} L(tR + \tilde{\phi}, R + \dot{\tilde{\phi}}) dt + \frac{1}{T_n} \int_{T_n}^{T_n - N[T_n/(km)]} L(tR + \tilde{\phi}, R + \dot{\tilde{\phi}}) dt, \]
which implies
\[ \lim_{n \to \infty} \int_0^{T_n} L(tR + \tilde{\phi}, R + \dot{\tilde{\phi}}) dt = \int_0^{km} L(sR + \phi, R + \dot{\phi}) ds. \]

We deduce \( \delta + \bar{L}(R) > \Phi(R) \) and, since \( \delta > 0 \) is arbitrary, we conclude the proof.

Remark 2.11. Thus, if \( L \) satisfies the assumptions of Lemma 2.8 we have \( \bar{L}(R) = \Phi(R) \) for all \( L_\mathbb{Q}(X) \). Before concluding that this is, in fact, true on the whole \( L^p(X) \), we need one more lemma.

Lemma 2.12. If \( \xi_n \to \xi \) in \( L^p(X) \) and \( \{\xi_n\}_{n=1}^\infty \subset L_\mathbb{Q}(X) \), then
\[ (2.25) \quad \bar{L}(\xi) \leq \liminf_{n \to \infty} \bar{L}(\xi_n). \]
Furthermore, there exists \( \xi_n \to \xi \) with \( \{\xi_n\}_{n=1}^\infty \subset L_\mathbb{Q}(X) \), such that
\[ (2.26) \quad \bar{L}(\xi) = \lim_{n \to \infty} \bar{L}(\xi_n). \]

Proof: Note that (2.25) follows trivially from the previous lemmas and the lower semicontinuity of \( \Phi \). To prove (2.26), we start by selecting a sequence \( T_n \to \infty \) and a sequence \( \phi_n \in W_0^{1,p}(0, T_n; L^p(X)) \) such that
\[ (2.27) \quad \bar{L}(\xi) = \lim_{n \to \infty} \int_0^{T_n} L(t\xi + \phi_n, \xi + \dot{\phi}_n) dt. \]
Next, as in [7], we let \( A_n = [T_n] + 2, s_n = 1 - (T_n - [T_n])/2 \). Then take \( \zeta_n \in C_c(X; \mathbb{R}^d) \) such that \( ||\zeta_n - \xi|| < 1/(2A_n) \). Define the vector field \([\zeta_n]_e \) by
\[ [y]_e,i = 2k \text{ whenever } 2k - 1 \leq y_i < 2k + 1 \text{ for } i = 1, ..., d, \]
and let \( \xi_n := [2A_n\xi_n]/(2A_n) \). Note that

\[
\|\xi_n - \xi\| < \frac{\sqrt{d}}{2A_n}
\]

and each component of \( \xi_n \) is an integer divided by \( A_n \).

Now let \( \psi_n \in W^{1,p}(0, A_n; L^p(X)) \) be given by

\[
\psi_n(s, x) = \begin{cases} 
0, & \text{if } 0 \leq s \leq s_n, \ x \in X \\
(s - s_n)\xi(x) + \phi_n(s - s_n, x), & \text{if } s_n \leq s \leq s_n + T_n, \ x \in X \\
\frac{s - s_n - T_n}{A_n} [A_n\xi_n(x) - T_n\xi(x)] + T_n\xi(x), & \text{if } s_n + T_n \leq s \leq A_n, \ x \in X.
\end{cases}
\]

Observe that \( \psi_n - s\xi_n \in W^{1,p}(0, T_n; L^p(X)) \) and

\[
\|\psi_n(t, \cdot)\| \leq \frac{1}{s_n}\|A_n\xi_n - T_n\xi\| \leq 2\|A_n\xi_n - T_n\xi\| \\
\leq 2A_n\|\xi_n - \xi\| + 2(A_n - T_n)\|\xi\| \leq C
\]

for all \( s_n + T_n \leq t \leq A_n \). Consequently,

\[
\int_0^{A_n} L(\psi_n, \dot{\psi}_n)dt \leq \int_0^{T_n+s_n} L(\psi_n, \dot{\psi}_n)dt + \int_{s_n+T_n}^{A_n} L(\psi_n, \dot{\psi}_n)dt \\
\leq \int_0^{T_n} L(t\xi + \phi_n, \xi + \dot{\phi}_n)dt + C(A_n - s_n - T_n),
\]

which implies

\[
\int_0^{A_n} L(\psi_n, \dot{\psi}_n)dt \leq \frac{T_n}{A_n} \int_0^{T_n} L(\phi_n, \dot{\phi}_n)dt + C\frac{A_n - s_n - T_n}{A_n}
\]

and so

\[
\limsup_{n \to \infty} \int_0^{A_n} L(\psi_n, \dot{\psi}_n)dt \leq \bar{L}(\xi).
\]

But by Remark 2.1

\[
\int_0^{A_n} L(\psi_n, \dot{\psi}_n)dt \geq \bar{L}(\xi_n), \text{ for all } n.
\]

This, together with (2.25) and (2.29) yields (2.26).

Later we shall need the continuity of the map \( \bar{L} \) with respect to the strong \( L^p(X) \) topology. This is obtained as a consequence of its convexity.

**Proposition 2.13.** The map \( \bar{L} \) is convex.

**Proof:** Fix \( Q_1, Q_2 \in L_1(X), \lambda \in [0, 1/2] \cap \mathbb{Q} \) and put \( Q_3 := \lambda Q_1 + (1 - \lambda)Q_2 \). Take \( \delta > 0 \) arbitrary. Due to Lemma 2.9, there exists \( T_0 > 0 \) such that

\[
\left| \inf_{\phi \in H_0(0, T)} \int_0^T L(tQ_1 + \phi, Q_1 + \dot{\phi})dt - L(Q_1) \right| \leq \frac{\delta}{2}
\]

for \( i = 1, 2, 3 \) and \( T \geq T_0 \). Next, consider \( T_1 \geq T_0 \) such that \( T_1 Q_1 \in L_1(X) \) and place \( T_2 := (-1 + 1/\lambda)T_1 \geq T_0 \). We can find \( \phi_i \in H_0(0, T_1) \) such that \( \phi_i(0, \cdot) = 0, \phi_i(T_1, \cdot) = T_1Q_i, \) and

\[
\int_0^{T_1} L(\phi_i, \dot{\phi}_i)dt \leq \bar{L}(Q_i) + \delta
\]
for $i = 1, 2$. Define $\tilde{\phi}$ to be the $\mathcal{H}_{(0,T_1+T_2)}$ map identically equal to $\phi_1$ on $[0,T_1] \times X$ and to $T_1Q_1 + \phi_2(t-T_1, \cdot )$ on $[T_1,T_1+T_2] \times X$. We obviously have $\tilde{\phi}(0, \cdot ) = 0$ and $\tilde{\phi}(T_1 + T_2, \cdot ) = (T_1 + T_2)Q_3$. Since

$$\int_0^{T_1+T_2} L(\tilde{\phi}, \dot{\tilde{\phi}})dt = \int_0^{T_1} L(\phi_1, \dot{\phi}_1)dt + \int_0^{T_2} L(\phi_2, \dot{\phi}_2)dt$$

and $\delta > 0$ is arbitrary, one obtains

$$\bar{L}(Q_3) \leq \lambda \bar{L}(Q_1) + (1 - \lambda)\bar{L}(Q_2).$$

We conclude by Lemma 2.12.

\section{2.5. The $I$-limit}

It is obvious from its definition that $\bar{L}$ is locally bounded. Therefore, its convexity implies the following:

\begin{corollary}
The map $\bar{L}$ is locally Lipschitz on $L^p(X)$.
\end{corollary}

So far, we have proved that

\begin{equation}
F(\phi, \Omega) = \int_\Omega \bar{L}(\phi)dt
\end{equation}

for all $\phi(t,x) = t\xi(x) + u(x)$ (the addition of $u$ is allowed due to the second equality in (2.21)), where $\xi$, $u \in L^p(X)$. In general, we have

\begin{theorem}
The equality (2.30) holds for every $\phi \in W^{1,p}_{loc}(\mathbb{R}; L^p(X)).$
\end{theorem}

The proof of this theorem is exactly as in [7], provided that we clarify some issues related to the infinite dimensionality of our setting. This is what we do next.

\begin{proof}[Proof of Theorem 2.15]
Due to the additivity property of $F$, we conclude (2.30) is also true for piecewise affine functions. To apply Egoroff’s Theorem and obtain

\begin{equation}
F(\phi, \Omega) \leq \int_\Omega \bar{L}(\phi)dt
\end{equation}

just as in [7], we only need to show that there exists a sequence of affine functions $\phi_n$ that converges strongly in $\mathcal{H}_\Omega$ to $\phi$. We refer to subsection 2.2.

For the reverse inequality, take any $\phi \in \mathcal{H}_{(a,b)}$ and recall that (see, e.g., [13]), if $\tilde{\phi}$ is the extension of $\phi$ to $\mathbb{R} \times X$ by $\tilde{\phi}(t, \cdot ) = \phi(a, \cdot )$ if $t < a$ and $\tilde{\phi}(t, \cdot ) = \phi(b, \cdot )$ if $t > b$, then the function

$$\alpha_t(s) = \begin{cases} \|\frac{\phi(t+s, \cdot ) - \phi(t)}{s} - \tilde{\phi}(t, \cdot )\|, & \text{if } s \neq 0 \\ 0, & \text{if } s = 0 \end{cases}$$

is continuous on $\mathbb{R}$ for a.e. $t \in (a,b)$. In particular, for a.e. $t_0 \in \mathbb{R}$,

\begin{align*}
0 &= \lim_{r \to 0^+} \frac{1}{r} \int_{-r}^{r} \|\frac{\phi(t_0 + s, \cdot ) - \phi(t_0)}{s} - \tilde{\phi}(t_0, \cdot )\|^{p} ds \\
&= \lim_{r \to 0^+} \int_{t_0-1}^{t_0+1} \frac{1}{|t-t_0|^{p}} \|\frac{\phi(t_0 + r(t-t_0), \cdot ) - \phi(t_0)}{r} - (t-t_0)\tilde{\phi}(t_0, \cdot )\|^{p} dt.
\end{align*}

We deduce that

$$\lim_{r \to 0^+} \int_{t_0-1}^{t_0+1} \|\frac{\phi(t_0 + r(t-t_0), \cdot ) - \phi(t_0)}{r} - (t-t_0)\tilde{\phi}(t_0, \cdot )\|^{p} dt = 0,$$

i.e., as $r \to 0^+$,

$$\frac{\phi(t_0 + r(t-t_0), \cdot ) - \tilde{\phi}(t_0, \cdot )}{r} - (t-t_0)\tilde{\phi}(t_0, \cdot ) \to 0$$

in $L^p((t_0-1, t_0+1) \times X)$.

The rest of the argument follows exactly as in [7].

\qed
3. Hamilton-Jacobi Equations on $\mathcal{P}_p(\mathbb{R})$: Homogenization

We fix $T > 0$, $p > 1$ and set $X = (0, 1)$. We denote by $\| \cdot \|$ the standard $L^p(0, 1)$ norm. Recall that $\mathcal{H} = AC^p(0, T; L^p(0, 1))$. If $\mu \in \mathcal{P}_p(\mathbb{R})$, $L^p(\mu)$ stands for the set of $v : \mathbb{R} \to \mathbb{R}$ that are $\mu$-measurable and such that $\|v\|_{L^p}^p := \int_{\mathbb{R}} |v|^p \, d\mu$ is finite. We denote by $\mathcal{S}$ the set $AC^p(0, T; \mathcal{P}_p(\mathbb{R}))$ of $p$-absolutely continuous paths on the Wasserstein space and denote by $W_p$ the $p$-Wasserstein distance. Some useful properties of the element of $\mathcal{P}(\mathbb{R})$ are recalled in the appendix (cf. subsection 7.1). We endow $\mathcal{S}$ with the following topology.

**Definition 3.1.** We say that $\{\sigma^n\}_n \subset \mathcal{S}$ $\tau_w$-converges to $\sigma$ in $\mathcal{S}$ if

\[ (3.1) \quad \lim_{n \to \infty} \int_0^T W_1(\sigma^n, \sigma) \, dt = 0 \]

\[ (3.2) \quad \sup_{n \in \mathbb{N}} \int_0^T \left( W_p(\sigma^n, \delta_0) + |(\sigma^n)'|_{L^p}^p \right) \, dt < \infty. \]

**Remark 3.2.** When $\phi$ is a flow associated to $\sigma$, $\nu_\phi = v_{\phi} \cap \phi_\phi$ is a flow associated to $\sigma$. Suppose $\phi : (0, 1) \to \mathbb{R}$ is the unique monotone nondecreasing, left continuous map such that $\phi_{\tau} v_{\phi} = \sigma_t$ then $\phi \in \mathcal{H}$ and $\nu_\phi = v_{\phi} \cap \phi_\phi$ for almost every $t \in (0, T)$ (cf. e.g. [13]). We say that $\phi$ is a flow associated to $\sigma$.

**Properties of the weak topology $\tau$.**

**Remark 3.3.** Let $A_n : (0, 1) \to \mathbb{R}$ be a sequence of monotone nondecreasing functions for which there exists $C > 0$ such that $\|A_n\|_{L^p(0, 1)} \leq C$ for all $n \in \mathbb{N}$. Suppose $A : (0, 1) \to \mathbb{R}$ is monotone nondecreasing. Then, for each $r \in (0, 1/2)$, $\{A_n\}_n$ is bounded in $L^\infty((r, 1-r))$ by a constant depending only on $C$ and $r$. Furthermore, the following are equivalent

(i) $\{A_n\}_n$ converges to $A$ in $L^1((0, 1))$.

(ii) $\{A_n\}_n$ converges to $A$ locally in $L^p((0, 1))$.

**Proposition 3.4.** Let $\{\sigma^n\}_n \subset \mathcal{S}$ and let $v^n$ be a velocity for $\sigma^n$. Let $\sigma \in \mathcal{S}$ and suppose that $v$ is a velocity for $\sigma$. Suppose $\phi^n : (0, 1) \to \mathbb{R}$ is the unique monotone nondecreasing, left continuous map such that $\phi^n_{\tau} v^n = \sigma^n_t$. Similarly, let $\phi_t$ be the unique monotone nondecreasing, left continuous map such that $\phi_{\tau} v_0 = \sigma_t$. Then the following assertions are equivalent:

\[ (3.3) \quad \{\sigma^n\}_n \quad \tau_w - \text{converges to } \sigma \text{ in } \mathcal{S} \]

\[ (3.4) \quad \{\phi^n\}_n \quad \tau - \text{converges to } \phi \text{ in } \mathcal{H}. \]

**Proof:** Recall that $\phi^n \in \mathcal{H}$ is equivalent to $\sigma^n \in \mathcal{S}$ and, as pointed in Remark 3.2, $\phi^n_t = v^n_t \circ \phi^n_t$ for almost every $t \in (0, T)$. Hence,

\[ (3.5) \quad \int_0^T \left( W_p(\sigma^n, \delta_0) + |(\sigma^n)'|_{L^p}^p \right) \, dt = \int_0^T \left( \|\phi^n_t\|^p + \|\phi^n_t\|^p \right) \, dt. \]

and so, $\{\phi^n\}_n$ and $\{\sigma^n\}_n$ are bounded in $L^p((0, 0, T) \times (0, 1))$ if and only if (3.2) holds. It remains to show that under (3.2), (3.1) holds if and only if $\phi_n$ converges locally to $\phi$ in $L^p((0, T) \times (0, 1))$. Let $C$ be such that $\|W_p(\sigma^n, \delta_0)\|_{L^p} \leq C$.

We assume in the sequel that (3.2) holds. By Remark 7.1 every subsequence of $\{\sigma^n\}_n$ admits a subsequence $\{\sigma^{n_k}\}_k$ such that $\{\sigma^{n_k}\}_k$ converges to $\sigma$ in $\mathcal{P}_p(\mathbb{R}^d)$ for every $t \in [0, T]$ and every $1 \leq q < p$. The limit being independent of the subsequence, we conclude that $\{\sigma^{n_k}\}_k$ converges to $\sigma$ in $\mathcal{P}_p(\mathbb{R}^d)$ for every $t \in [0, T]$ and every $1 \leq q < p$. As $W_1(\sigma^n, \sigma) = \|\phi^n - \phi\|_{L^1(0, 1)}$, we conclude that $\{\phi^n\}_n$ converges to $\phi$ in $L^1((0, 1))$ for all $t \in (0, T)$. By Remark 3.3 and the fact that $W_p(\sigma^n, \delta_0) = \|\phi^n_t\|^p \leq C$ for all $n$, if $r \in (0, 1/2)$ then $\{M^n_{\phi^n}\}_n$ converges to $\phi_t$ in $L^p((0, T) \times (0, 1-r))$ for all $t \in [0, T]$ and $\{\phi^n\}_n$ is bounded in $L^\infty((0, T) \times (r, 1-r))$. We use the
Lebesgue dominated convergence theorem to conclude that \( \{ \phi^n \}_n \) converges to \( \phi \) in \( L^p((0, T) \times (r, 1 - r)) \). Thus, \( \{ \phi^n \}_n \) converges locally to \( \phi \) in \( L^p((0, T) \times (0, 1)) \).

Conversely, suppose that \( \{ \phi^n \}_n \) converges locally to \( \phi \) in \( L^p((0, T) \times (0, 1)) \). Let \( C \) be the supremum over \( n \) of the expressions in (3.5). Let \( r \in (0, 1/2) \), set \( S = (0, T) \setminus (r, 1 - r) \) and denote by \( S^c \) the complement of \( S \) in \( (0, T) \times (0, 1) \). We have

\[
\int_0^T W_1(\sigma^n _t, \sigma_t)dt \leq \| \phi^n - \phi \|_{L^p(S)}(L^2(S))^{\frac{1}{p}} + \| \phi^n - \phi \|_{L^p(S^c)}(L^2(S^c))^{\frac{1}{p}}
\]

\[
\leq \| \phi^n - \phi \|_{L^p(S)}(L^2(S))^{\frac{1}{p}} + 2C^{\frac{1}{p}}(2rT)^{\frac{1}{p'}}.
\]

This proves that \( \limsup_{n \to \infty} \int_0^T W_1(\sigma^n _t, \sigma_t)dt \leq 2C^{\frac{1}{p}}(2rT)^{\frac{1}{p'}}. \) Since \( r \in (0, 1/2) \) is arbitrary, we conclude that (3.1) holds.

Lemma 3.5. Let \( \{ \sigma \} \cup \{ \sigma^n \}_n \subset S \) and let \( v^n \) (resp. \( v \)) be the velocity for \( \sigma^n \) (resp. \( \sigma \)). Suppose \( \phi^n : (0, 1) \to \mathbb{R} \) (resp. \( \phi : (0, 1) \to \mathbb{R} \)) is the unique monotone nondecreasing left continuous map such that \( \phi^n_{\#} v^n = \sigma^n \) (resp. \( \phi_{\#} v = \sigma \)). Let \( \tilde{\phi} \) (resp. \( \tilde{\phi} \)) be a flow associated to \( \sigma^n \) (resp. \( \sigma \)). Suppose \( \{ \tilde{\phi}_n \}_n \) \( \tau \)-converges to \( \tilde{\phi} \) \( \tau \)-converges to \( \phi \).

Proof: As \( \phi^n \) and \( \tilde{\phi} \) are flows associated to \( \sigma^n \) we may substitute \( \phi^n \) by \( \tilde{\phi} \) in (3.5) to obtain that \( \{ \tilde{\phi}_n \}_n \) and \( \{ \tilde{\phi}_n \}_n \) are bounded in \( L^p((0, T) \times (0, 1)) \). We use that \( \| \phi^n - \phi_n \|_{L^1((0, 1))} \leq \| \tilde{\phi} - \tilde{\phi} \|_{L^1((0, 1))} \) (cf. e.g. [13]) to conclude that

\[
\lim_{n \to \infty} \int_0^T W_1(\sigma^n _t, \sigma_t)dt = \lim_{n \to \infty} \int_{(0,T) \times (0,1)} |\phi^n - \phi|d\sigma dt \leq \lim_{n \to \infty} \int_{(0,T) \times (0,1)} |\tilde{\phi} - \tilde{\phi}|d\sigma dt = 0.
\]

We use Proposition 3.4 to conclude that \( \{ \tilde{\phi}_n \}_n \) \( \tau \)-converges to \( \phi \) in \( \mathcal{H} \).

3.2. Effective Lagrangian on \( \mathcal{P}_p(\mathbb{R}) \). As in section 2 we assume that \( L \) is a Lagrangian on \( L^p(X) \times L^p(X) \) satisfying the growth conditions (2.1) and the periodicity condition (2.2). We further assume the following invariance property: if \( M, M' \in L^p(X) \) satisfy \( M_{\#} \nu_0 = M'_{\#} \nu_0 =: \mu \) and \( v \in L^p(\mu) \) then

\[
(3.6) \quad L(M, v \circ M) = L(M, v \circ M').
\]

We define the Lagrangian \( \mathcal{L} \) by

\[
(3.7) \quad \mathcal{L}(\mu, v) = L(M, v \circ M).
\]

We denote by \( \mathcal{T}\mathcal{P}_p(\mathbb{R}) \) the set of pairs \((\mu, v)\) such that \( \mu \in \mathcal{P}_p(\mathbb{R}) \) and \( v \in L^p(\mu) \). In [2] \( \mathcal{T}\mathcal{P}_p(\mathbb{R}) \) is referred to as the tangent bundle of \( \mathcal{P}_p(\mathbb{R}) \). For \( \sigma \in \mathcal{S}, v \) the velocity for \( \sigma \) and \( \epsilon > 0 \) we define

\[
\mathcal{F}_\epsilon(\sigma) = \int_0^T \mathcal{L} \left( D^{1/\epsilon}_\# /\sigma_t, v_t \circ D^{1/\epsilon}_\# \right) dt.
\]

We do not display the dependence of \( \mathcal{F}_\epsilon \) on \( v \), as the velocity of minimal norm for \( \sigma \) is univoquely determined. We set

\[
\mathcal{F}(\sigma) = \int_0^T \mathcal{L}(v_t \circ \phi_t) dt,
\]

where \( \phi_t \) is the unique monotone nondecreasing left continuous map that pushes \( \nu_0 \) forward to \( \sigma_t \).

Theorem 3.6. The family \( \{ \mathcal{F}_\epsilon \}_\epsilon \) \( \Gamma(\tau) \)-converges to \( \mathcal{F} \) as \( \epsilon \) tends to 0.
Let us fix $\Lambda$ so that $(3.10)$ satisfies $(3.6)$. Furthermore, $\bar{\phi}$ and consider the extension to Homogenization for Hamilton-Jacobi equations in the Wasserstein space.

3.3. Homogenization for Hamilton-Jacobi equations in the Wasserstein space. Let us fix $\Lambda > 0$ and consider the extension to $T\mathcal{P}_2(\mathbb{R})$ of the classical mechanical Lagrangian, i.e.,

$$L(\mu, \xi) = \Lambda \|\xi\|^2 - W(\mu) \text{ for all } (\mu, \xi) \in T\mathcal{P}_2(\mathbb{R}),$$

which corresponds to

$$(3.10) \quad L(M, N) = \Lambda \|N\|^2 - W(M) \text{ for all } (M, N) \in [L^2(X)]^2$$

via $L(M_\# \nu_0, \xi) = L(M, \xi \circ M)$ for $M \in L^2(X)$. We hope the reader will be convinced that the results proved in this section hold even if $L$ is not necessarily of this form but simply satisfies the assumptions from the previous section and, additionally, it is continuous in both variables and nondecreasing in $\|N\|$. As an application of Theorem 3.6, below we provide a proof of the fact that the viscosity solutions given by the backward Lax-Oleinik formula (see [14]) for the time-dependent Hamilton-Jacobi equations corresponding to the Lagrangians $\mathcal{L}^\epsilon$ (defined below) converge to the backward Lax-Oleinik solution for

$$(3.11) \quad \partial_t U(t, \mu) + \mathcal{H}(\mu, \nabla_W U(t, \mu)) = 0, \quad U(0, \mu) = U_0(\mu),$$

where $\nabla_W$ denotes the Wasserstein gradient (cf. [14]) and

$$(3.12) \quad \mathcal{H}(\mu, \zeta) = \sup_{\zeta \in L^2(X)} \{\langle \zeta, \zeta \rangle_{L^2(X)} - \mathcal{L}(\mu, \xi)\} \text{ for all } \zeta \in L^2(X)$$

is the Legendre transform of $\mathcal{L}$. The latter can be defined by (3.7) due to the fact that the Lagrangian in (3.10) satisfies (3.6). Furthermore, $\mathcal{L}$ satisfies the necessary conditions for the existence theory in [14] to apply to (3.11). We assume

$$(3.13) \quad u_0 : L^2(X) \to \mathbb{R} \text{ is bounded below, } \ W \text{ is bounded, nonpositive and 1-periodic, and satisfies, for some continuous nondecreasing } \omega : \mathbb{R} \to \mathbb{R} \text{ such that } \omega(0) = 0,$$

$$W(M_1) - W(M_2) \leq \omega(\mathcal{L}^\epsilon(\{M_1 \neq M_2\}))$$

for all $M_1, M_2 \in L^2(X)$. Thus, the conditions (2.1), (2.22), (2.23) on $L$ are all satisfied. We also assume

$$(3.15) \quad W \text{ is } L^2_{loc}(X)\text{-upper semicontinuous, } u_0 \text{ is } L^2_{loc}(X)\text{-lower semicontinuous.}$$

Denote by $\mathcal{M}$ the set of all monotone nondecreasing functions in $L^2(X)$. Let $L^\epsilon(M, N) := L(M/\epsilon, N)$ for $\epsilon > 0$ so that

$$(3.5) \quad L^\epsilon(\mu, \xi) := L(D^{1/\epsilon}_\# \mu, \xi \circ D^\epsilon).$$
The following minima exist and are equal (cf. [13]):

\[ U^*(t, \mu) = \min_{\sigma \in \mathcal{C}_t(M_0, M)} \left\{ U_0(\sigma(0)) + \int_0^T L^*(\sigma, \dot{\sigma})d\tau \right\} \]

\[ = \min_{\phi \in \mathcal{C}_t(M_0, M)} \left\{ U_0(\phi_0) + \int_0^T L^*(\phi, \dot{\phi})d\tau \right\} = U_0(M^*(0)) + \int_0^T L^*(M^*(s), \dot{M}^*(s))d\tau, \]

where \( M_\mu \) is the monotone rearrangement of \( \nu_0 \) into \( \mu \) and \( \mathcal{C}_t(M; N; \mathcal{M}) \) is the set of all curves \( \phi \in H^1(0, T; \mathcal{M}) \) such that \( \phi_0 = M, \phi_t = N \). According to [14], \( U^* \) is a viscosity solution for

\[
\partial_t U(t, \mu) + H^*(\mu, \nabla_W U(t, \mu)) = 0, \quad U(0, \mu) = U_0(\mu),
\]

where the Legendre transform \( H^* \) of \( L^* \) is defined as in (3.12) with \( L^* \) replacing \( \tilde{L} \). Below we prove the convergence of these viscosity solutions to the corresponding viscosity solution for (3.11).

**Proposition 3.7.** Assume \( u_0 \) is continuous with respect to the \( L^2_{loc}(X) \) strong topology and let \( M_T \in \mathcal{M} \). Let \( \epsilon_n \downarrow 0 \) and \( M^n \in \mathcal{C}(\epsilon, M_T) \) be a minimizer for

\[ \left\{ A^0(S) := u_0(S(0)) + \int_0^T L^n(S(s), \dot{S}(s))d\tau : S \in \mathcal{C}(\epsilon, M_T) \right\}. \]

Then, possibly up to a subsequence, \( M^n \rightharpoonup M \) for some \( M \in H^1(0, T; \mathcal{M}) \) which is a minimizer for

\[ \left\{ A(S) := u_0(S(0)) + \int_0^T L(S(s))d\tau : S \in \mathcal{C}(\epsilon, M_T) \right\}. \]

Furthermore, as \( n \to \infty \),

\[
\min_{S \in \mathcal{C}_T(\epsilon, M_T)} \left\{ u_0(S(0)) + \int_0^T L^n(S(s), \dot{S}(s))d\tau \right\} \to \min_{S \in \mathcal{C}_T(\epsilon, M_T)} \left\{ u_0(S(0)) + \int_0^T L(S(s))d\tau \right\}.
\]

**Proof:** Recall that, in fact, \( M^n \in \mathcal{C}(M_0, M_T; \mathcal{M}) \). Since \( M^n \) is clearly bounded in \( L^2((0, T) \times X) \), we deduce that a subsequence \( \{M^n\}_n \) converges in the \( \tau \) topology to some \( M \in \mathcal{C}_T(M_0, M_T; \mathcal{M}) \). But we know that the actions of \( L^n \) \( \tau \)-converge with respect to the \( \tau \)-topology to the action of \( L \). All that is left to show is the \( \Gamma(\tau) \)-convergence of \( A^n \) to \( A \). According to Proposition 2.4 in [7], it is enough to prove that the map \( S \to u_0(S(0)) \) is continuous on \( H^1(0, T; L^2(X)) \) with respect to the \( \tau \)-topology. But \( S_n \rightharpoonup S \) implies (see [13]) \( S_n(s) \to S(s) \) in \( L^2_{loc}(X) \)-strong for all \( s \in [0, T] \). In particular, \( S_n(0) \to S(0) \) in \( L^2_{loc}(X) \), so \( u_0(S_n(0)) \to u_0(S(0)) \).

\[ \Box \]

4. Applications to the 1D Pressureless Gas Dynamics

Note that \( L_1 \) and \( L_2 \) defined in the Introduction satisfy (2.1), (2.22) and (2.23).

4.1. Some special Lagrangians. We shall prove

**Proposition 4.1.** One has

\[
\bar{L}_1(\xi) = \int_X l_1 \circ \xi \, dx,
\]

where \( l_1 \) is the effective Lagrangian associated to \( l_1(x, v) = v^2/2 - V(x) \).

**Proof:** If \( M \in H^1_0(0, T; L^2(X)) \), then \( M(\cdot, x) \in H^1_0(0, T) \) for a.e. \( x \in X \). Since

\[
\int_0^T L_1(M + tx, \dot{M} + \xi)dt = \int_X \int_0^T \left\{ \frac{1}{2} |\dot{M}(t, x) + \xi(x)|^2 - V(M(t, x) + \xi(x)) \right\}dt \, dx,
\]

we have

\[
\bar{L}_1(\xi) = \int_X l_1 \circ \xi \, dx.
\]

\[ \Box \]
we infer
\[ \inf_M \int_0^T L_1(M + t \xi, \dot{M} + \xi) dt \geq \int_X \inf_M \int_0^T \left\{ \frac{1}{2} |\dot{M}(t, x) + \xi(t)|^2 - V(M(t, x) + t \xi(x)) \right\} dt dx. \]
Thus, Fatou's Lemma yields
\[ \bar{L}_1(\xi) \geq \int_X \bar{l}_1 \circ \xi dx. \]
To prove the opposite inequality, we take a sequence of discrete \( \xi_n \to \xi \) in \( L^2(X) \) and a.e. such that \( \xi_n \) is constantly \( \xi^1_n \in \mathbb{R} \) on \((c_{i-1}, c_i)\) (regular partition of \(X\), \(i = 1, \ldots, n\). Clearly,
\[ \inf_M \int_0^T L_1(M + t \xi_n, \dot{M} + \xi_n) dt \leq \frac{1}{n} \inf_{r \in H^1(0,T;\mathbb{R}^n)} \sum_{i=1}^n \int_0^T l_1(r_i + t \xi^i_n, \dot{r}_i + \xi^i_n) dt \]
\[ = \frac{1}{n} \sum_{i=1}^n \inf_{r \in H^1(0,T;\mathbb{R})} \int_0^T l_1(r + t \xi^i_n, \dot{r} + \xi^i_n) dt. \]
Passing to \(\liminf\) as \(T \to \infty\), we obtain
\[ \bar{L}_1(\xi_n) \leq \int_X \bar{l}_1 \circ \xi_n dx. \]
Due to the continuity of \( \bar{L}_1 \) over \( L^2(X) \) and of \( \bar{l}_1 \) over \( \mathbb{R} \), we can pass to the limit as \(n \to \infty\) to conclude the proof.

We introduce the Lagrangian \( L_4 : [L^2(X)]^2 \times [L^2(X)]^2 \to \mathbb{R} \) given by
\[ L_4(M_1, M_2, N_1, N_2) = \frac{1}{4} \|N_1\|^2 + \frac{1}{4} \|N_2\|^2 - \int_X V(M_1(x) - M_2(y)) dy dx. \]
As we shall see later, this corresponds to a system of two interacting species of particles. Arguing as in the proof of Proposition 4.1, one can show that
\[ \bar{L}_4(\xi_1, \xi_2) = \int_{X^2} \bar{l}_2(\xi_1(x), \xi_2(y)) dy dx \]
for any \( \xi_1, \xi_2 \in L^2(X) \).

In the case of the two-particle (same species) interaction Lagrangian, we can rewrite it as
\[ L_2(M, N) = \int_X \left\{ \frac{1}{4} |N(x)|^2 + \frac{1}{4} |N(y)|^2 - V(M(x) - M(y)) \right\} dy dx. \]
Thus, the natural question is whether \( \bar{L}_2(\xi) \) coincides with \( \bar{L}_4(\xi, \xi) = \int_{X^2} \bar{l}_2(\xi(x), \xi(y)) dy dx \), where \( \bar{l}_2 \) is the Lagrangian on \( \mathbb{R}^2 \times \mathbb{R}^2 \) given by \( \bar{l}_2(x, v) = |v|^2/4 - V(x_1 - x_2) \). The answer is, in general, negative. The following lemma will be used in the sequel.

**Lemma 4.2.** Let \( T > 0 \) and \( \alpha \in \mathbb{R} \) such that \( \alpha T \in \mathbb{Z} \). Then there is a unique solution for \( \ddot{r} = -4V'(r) \) with boundary conditions \( r(0) = 0 \), \( r(T) = \alpha T \). This solution is precisely the unique minimizer
\[ \arg\min \left\{ \int_0^T l(r, \dot{r}) dt : r \in H^1(0, T), \ r(0) = 0, \ r(T) = \alpha T \right\}, \]
where
\[ l : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \ l(x, v) = v^2/8 - V(x). \]

**Proof:** It is easy to prove that the minimizer exists, it is of class \( C^1 \) and its Euler-Lagrange equation is exactly the ODE above. Let us assume that \( \alpha > 0 \) (the other cases are handled similarly). Any solution satisfies
\[ \frac{1}{8} |\dot{r}(t)|^2 + V(r(t)) = \frac{1}{8} |r(0)|^2 \]
because \(r(0) = 0\) and \(V(0) = 0\). Due to \(V \leq 0\), \(\alpha > 0\) and the continuity of \(\dot{r}\), we infer
\[
\dot{r}(t) = \sqrt{m_0^2 - 8V(r(t))},
\]
where \(m_0 = \dot{r}(0) > 0\).

Indeed, note that \(\dot{r}(0)\) cannot be nonpositive. If \(F(t) := \int_0^t (m_0^2 - 8V(s))^{-1/2}ds\), then \(F(r(t)) = t\) for all \(t \in [0, T]\). In particular, since \(r(T) = \alpha T\) is a positive integer and \(F'\) is periodic, then \(1/\alpha = \int_0^1 (m_0^2 - 8V(s))^{-1/2}ds\), so \(\dot{r}(0)\) is uniquely determined. So, by uniqueness for the initial-value problem, we conclude the proof. \(\square\)

We are now ready to justify the following somewhat surprising result.

**Proposition 4.3.** One has

\[
\bar{l}_2(a, b) = \frac{1}{8}(a + b)^2 + \bar{l}(a - b) \quad \text{for all } a, b \in \mathbb{R}.
\]

**Proof:** Let us consider minimizing
\[
\int_0^T l_2(r, \dot{r}) dt = \int_0^T \left( \frac{1}{4} |\dot{r}_1|^2 + \frac{1}{4} |\dot{r}_2|^2 - V(r_1 - r_2) \right) dt
\]
over \(H^1(0, T; \mathbb{R}^2)\) with \(r(0) = 0\), \(r(T) = Tc \in \mathbb{Z}^2\). Clearly, the minimizer \(r\) exists, is in \(C^3\), and it satisfies the Euler-Lagrange system
\[
\dot{r}_1 = -2V'(r_1 - r_2), \quad \dot{r}_2 = -2V'(r_2 - r_1) \quad \text{with } r(0) = 0, \ r(T) = Tc.
\]

Since \(V\) is even, by adding the two equations we infer \(\dot{r}_1 + \dot{r}_2 = 0\), so
\[
\dot{r}_1 = \dot{r}_2 = c_1 + c_2 \quad \text{on } [0, T].
\]

By subtracting the equations we get that \(r := r_1 - r_2\) satisfies \(\ddot{r} = -4V'(r)\). But \(r(0) = 0\) and \(r(T) = T(c_1 - c_2) =: Tc \in \mathbb{Z},\) so, according to Lemma 4.2, \(r\) is the unique minimizer of \(\int_0^T l(r, \dot{r}) dt\) with the prescribed endpoints. Therefore, due to (4.7),
\[
\min_{u(0) = 0, u(T) = Tc} \int_0^T l(u, \dot{u}) dt = \int_0^T \left( \frac{1}{8} |\dot{r}|^2 - V(r_1 - r_2) \right) dt = \min_{r(0) = 0, r(T) = Tc} \int_0^T l_2(r, \dot{r}) dt - \frac{1}{8}(c_1 + c_2)^2.
\]

We conclude that (4.6) is true for any \(a, b \in \mathbb{Q}\) by taking \(\lim\inf\) as \(T \uparrow \infty\). We finish the argument by using the continuity of \(\bar{l}\) and \(l_2\). \(\square\)

We next introduce a new one-dimensional Lagrangian \(\bar{\bar{l}}(x, v) = v^2/4 - V(x)\). Note that, if \(a \in \mathbb{R}\),
\[
\bar{\bar{l}}(a) = \lim_{T \to \infty} \min_{r(0) = 0, r(T) = aT} \int_0^T \left( \frac{1}{8} |\dot{r}|^2 + \frac{1}{8} |\ddot{r}|^2 - V(r) \right) dt \geq \frac{1}{8} a^2 + \bar{l}(a)
\]
because \(\int_0^T |\ddot{r}|^2 dt \geq a^2\). We are now ready to prove:

**Proposition 4.4.** (1) For every \(\xi \in L^2(X)\),
\[
\bar{L}_2(\xi) \geq \int_X \int_X \bar{l}_2(\xi(x), \xi(y)) dy dx.
\]

(2) Unless \(V \equiv 0\), there exists \(\xi \in L^2(X)\) such that
\[
\bar{L}_2(\xi) > \int_X \int_X \bar{l}_2(\xi(x), \xi(y)) dy dx.
\]
Proof: It is easy to see that $\bar{L}_4(\xi, \xi) \leq \bar{L}_2(\xi)$ for all $\xi \in L^2(X)$ (the infima are taken over a smaller class for $L_2$).

We shall prove (4.10) by contradiction. Assume it is false. Then, due to (4.9), we have that

\[
\bar{L}_2(\xi) = \int_{X^2} \bar{l}_2(\xi(x), \xi(y))dydx \text{ for all } \xi \in L^2(X).
\]

Let us take an arbitrary $M \in H^1(0, T; L^2(X))$ with $M(0, \cdot) = 0$, $M(T, x) = T\xi(x)$ and write

\[
\int_0^T \bar{L}_2(M, \dot{M})dt = \int_0^T \int_{X^2} \left( \frac{1}{2} \left| M(t, x) - \dot{M}(t, y) \right|^2 - V(M(t, x) - M(t, y)) \right)dydxdt
\]

\[
+ \frac{1}{2} \int_0^T \left( \int_X \dot{M}(t, x)dx \right)^2 dt 
\]

\[
\geq \int_{X^2} \min_{r(0) = 0, \dot{r}(T) = (\xi(x) - \xi(y))T} \left\{ \int_0^T \bar{l}(r, \dot{r})dt \right\}dydx + \frac{1}{2} \left( \int_X \xi dx \right)^2.
\]

Passing to $\liminf$ as $T \to \infty$ and using Fatou’s Lemma, we deduce

\[
\tilde{L}_2(\xi) \geq \int_{X^2} \bar{l}(\xi(x) - \xi(y))dydx + \frac{1}{2} \left( \int_X \xi dx \right)^2,
\]

which, together with (4.11) and Proposition 4.3, gives

\[
\frac{1}{8} \int_{X^2} |\xi(x) - \xi(y)|^2dydx \geq \int_{X^2} \left[ \bar{l}(\xi(x) - \xi(y)) - \bar{l}(\xi(x) - \xi(y)) \right]dydx
\]

for all $\xi \in L^2(X)$. Thus, $a^2/8 \geq \bar{l}(a) - \bar{l}(a)$ for all $a \in \mathbb{R}$. But this and (4.8) imply

\[
a^2/8 = \bar{l}(a) - \bar{l}(a) \text{ for all } a \in \mathbb{R}.
\]

By definition, we also have $\bar{l}(a) = \bar{l}(a\sqrt{2})$ for all $a \in \mathbb{R}$. So, (4.13) implies

\[
a^2/4 = \bar{l}(a\sqrt{2}) - \bar{l}(a) \text{ for all } a \in \mathbb{R}.
\]

The Legendre transform of $\bar{l}$ is $\bar{h}(x, p) = p^2 + V(x)$. Its effective Hamiltonian is computed in [17] as

\[
\bar{h}(p) = \begin{cases} 
0 & \text{if } |p| \leq \sqrt{-V} \\
\lambda & \text{if } |p| = \sqrt{-V + \lambda}
\end{cases}
\]

so, since $\bar{l}$ is its Legendre transform, its subgradient at 0, $\partial \bar{l}(0)$, has a maximal element $p_0 \geq \sqrt{-V} > 0$ unless $V \equiv 0$ (trivial case). If we divide (4.14) by $a\sqrt{2}$ for $a > 0$ and let $a \to 0^+$ we obtain $p_0 = p_0/\sqrt{2}$ which contradicts $p_0 > 0$.

\[\square\]

Remark 4.5. It is not clear whether equality in (4.12) holds for all $\xi \in L^2(X)$.

4.2. Homogenization for action minimizing solutions. Fix $T > 0$ and $\rho_0, \rho_T \in \mathcal{P}_2(\mathbb{R})$. One consequence of the analysis in the previous section is that if we consider action-minimizing solutions (shown to exist) of

\[
\begin{cases} 
\partial_t \rho' + \partial_x (\rho'v') = 0 \\
\partial_t (\rho'v') + \partial_x \left[ \rho'(v')^2 \right] = -\frac{1}{\varepsilon} \rho' A_i'
\end{cases}
\]

with prescribed endpoints $\rho'(0) = \rho_0$ and $\rho'(T) = \rho_T$, then the corresponding optimal maps $M_{\rho(\cdot)}$ converge in $L^2_{loc}(X)$ to $M(t)$, where $M \in H^1(0, T; L^2(X))$ minimizes the action $\int_0^T \bar{L}_i(M)dt$. Recall that here
Let \( L \) be the two-particle interaction Lagrangian respectively. Let \( \rho(t) := M(t)_{\#} \nu_0 \). It follows that \( \rho'(t, \cdot) \) converges to \( \rho(t, \cdot) \) in the \( p \)-Wasserstein distance for any \( 1 \leq p < 2 \). So far, we have not been able to identify the limiting system of equations satisfied by \( (\rho, v) \) in the case of the two-particle interaction Lagrangian \( L_2 \). We can only prove:

**Theorem 4.6.** Let \((\rho', v')\) be action-minimizing solutions for

\[
\begin{align*}
\frac{\partial}{\partial t} \rho' + \frac{\partial}{\partial x} (\rho' v') &= 0 \\
\frac{\partial}{\partial x} (\rho' v') + \frac{\partial}{\partial p} \left[ \rho' (v')^2 \right] &= -\frac{1}{2} \epsilon \rho' V'(\frac{x}{\epsilon})
\end{align*}
\]

(4.16)

with prescribed \( \rho'(0) = \rho_0 \) and \( \rho'(T) = \rho_T \). Then, the limiting \( \rho \) is the geodesic in the Wasserstein space connecting \( \rho_0 \) and \( \rho_T \). Thus, \((\rho, v)\) satisfies the pressureless Euler system

(4.17)

\[
\begin{align*}
\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} (\rho v) &= 0 \\
\frac{\partial}{\partial x} (\rho v) + \frac{\partial}{\partial p} (\rho v^2) &= 0.
\end{align*}
\]

**Proof:** Since we know that \( t \to M(t) \) minimizes

\[
\int_0^T L_1(M) dt = \int_0^T \int_X \tilde{L}_1(M(t,x)) dx dt,
\]

(equality due to Proposition 4.1) it is enough to show that \( \tilde{L}_1 \) is strictly convex to conclude that \( M(t,x) = (1-t/T)M_0(x) + (t/T)M_T(x) \). The Legendre transform of \( l_1 \) is \( h_1(x,p) = p^2/2 + V(x) \). Just as before we recall the computation in [17] to write down its effective Hamiltonian

\[
h_1(p) = \begin{cases} 
0 & \text{if } |p| \leq \langle \sqrt{-2V} \rangle \\
\lambda & \text{if } |p| = \langle \sqrt{-2V} + 2\lambda \rangle.
\end{cases}
\]

So,

\[
p = \sqrt{2} \int_0^1 \sqrt{h_1(p) - V(x)} dx \quad \text{for } p \geq \langle \sqrt{-2V} \rangle =: p_0.
\]

We infer that \( h_1 \) is differentiable at all \( p \) with \( p > p_0 \) and

\[
h_1'(p) = \frac{1}{\int_0^1 \sqrt{h_1(p) - V(x)} dx} \quad \text{for } p > p_0.
\]

Since \( h_1 \) is even, it is enough show that

\[
\lim_{p \to p_0^-} \int_0^1 \frac{dx}{\sqrt{h_1(p) - V(x)}} = \infty
\]

in order to conclude that \( h_1 \) is differentiable at \( \pm p_0 \). But \( V \) is \( C^2 \) and \( V(0) = V'(0) = 0 \), so there exists \( C > 0 \) such that \( \sqrt{-V(x)} \leq Cx \) on \([0, 1] \). This, combined with the fact that \( h_1(p) \downarrow 0 \) as \( p \to p_0^- \), yields that the limit in the last display is, indeed, \( \infty \). Since \( h_1 = 0 \) on \([-p_0, p_0] \), we obtain that \( h_1 \) is differentiable everywhere on \( \mathbb{R} \). Thus, \( \tilde{l}_1 \) is strictly convex.

\[
\square
\]

Likewise, \( l \) (for \( l \) defined in (4.5)) is strictly convex. In view of Proposition 4.3, we infer the strict convexity of \( l_2 \) and then, due to (4.3), the strict convexity of \( \tilde{L}_4 \). Thus, we may state the following true result.
Let $\rho_i^\epsilon, v_i^\epsilon$, $i = 1, 2,$ be action-minimizing solutions for

$$
\begin{align*}
\partial_t \rho_i^\epsilon + \partial_x (\rho_i^\epsilon v_i^\epsilon) &= 0, \\
\partial_t (\rho_i^\epsilon v_i^\epsilon) + \partial_x [\rho_i^\epsilon (v_i^\epsilon)^2] &= -1/\epsilon \rho_i^\epsilon A_i^\epsilon, \quad i \neq j
\end{align*}
\tag{4.18}
$$

with prescribed $\rho_i^\epsilon(0) = \rho_{i,0}$ and $\rho_i^\epsilon(T) = \rho_{i,T}$, $i = 1, 2$. Then, the limiting $\rho_i$ are geodesics in the Wasserstein space connecting $\rho_{i,0}$ and $\rho_{i,T}$. Thus, $(\rho_i, v_i)$ decouple to each satisfy the pressureless Euler system (4.17) for $i = 1, 2$.

To prove existence for the action minimizing solutions $(\rho_i^\epsilon, v_i^\epsilon)$ for fixed endpoints $\rho_i^\epsilon(0) = \rho_{i,0}$ and $\rho_i^\epsilon(T) = \rho_{i,T}$, we consider minimizing the action

$$
\int_0^T L_4 \left( \frac{M}{\epsilon}, M \right) dt
$$

over the set $M = (M_1, M_2) \in H^1(0, T; L^2(X))$ such that $M_1$ are prescribed and monotone nondecreasing at $0, T$. Then we adapt the proof (existence of minimizing paths) in [14] to get existence of minimizers. The corresponding Euler-Lagrange system is

$$
\begin{align*}
\tilde{M}_1(t, x) &= -\frac{2}{\epsilon} \int_X V' \left( \frac{M_1(t, x) - M_2(t, y)}{\epsilon} \right) dy \\
\tilde{M}_2(t, x) &= -\frac{2}{\epsilon} \int_X V' \left( \frac{M_2(t, x) - M_1(t, y)}{\epsilon} \right) dy
\end{align*}
\tag{4.19}
$$

and a minimizing pair $(M_1(t, \cdot), M_2(t, \cdot))$ consists of monotone nondecreasing maps in $L^2(X)$ for all $t \in [0, T]$. Let $\varphi_i \in C_1^1((0, T) \times X)$ and multiply the equations in the above system by $\varphi_i(t, M_i(t, x))$ for $i = 1, 2$ respectively. Then integrate on $[0, T] \times X$ by performing integration by parts in time for the left hand side. After that, we use Proposition 4.2 in [13] to obtain, upon setting $\rho_i^\epsilon(t, \cdot) = M_i(t, \cdot)\#\nu_0$, the formulation of (4.18) in the sense of distributions. To show some detail:

$$
\begin{align*}
\int_0^T \int_X \tilde{M}_i \varphi_i(t, M_i) dz dt &= -\int_0^T \int_X [\tilde{M}_i \partial_t \varphi_i(t, M_i) + |\tilde{M}_i|^2 \partial_x \varphi_i(t, M_i)] dz dt \\
&= -\int_0^T \int_X [v_i(t, M_i) \partial_t \varphi_i(t, M_i) + v_i(t, M_i)^2 \partial_x \varphi_i(t, M_i)] dz dt \\
&= -\int_0^T \int_\mathbb{R} [v_i(t, x) \partial_t \varphi_i(t, x) + v_i^2(t, x) \partial_x \varphi_i(t, x)] \rho_i(t, dx) dt.
\end{align*}
$$

But the $i^{th}$ equation in (4.19) also gives

$$
\begin{align*}
\int_0^T \int_X \tilde{M}_i \varphi_i(t, M_i) dz dt &= -\frac{2}{\epsilon} \int_0^T \int_{\mathbb{R}^2} V' \left( \frac{M_i(t, z) - M_j(t, z')}{\epsilon} \right) dz' dz dt \nonumber \\
&= -\frac{2}{\epsilon} \int_0^T \int_{\mathbb{R}^2} V' \left( \frac{x - y}{\epsilon} \right) \rho_j(t, dy) \rho_i(t, dx) dt,
\end{align*}
$$

where $i \neq j$. By equating the two we get the desired result (note that we dropped the $\epsilon$ superscripts). Note that the continuity equations hold automatically due to $\tilde{M}_i(t, \cdot) = v_i(t, M_i(t, \cdot))$ (again, see Proposition 4.2 in [13]).

5. $\Gamma$-limits on $\mathcal{P}_p(\mathbb{R}^d)$

Throughout this subsection $T > 0$ is prescribed and $\nu_0$ is the Lebesgue measure on $X = (0, 1)^d$. We assume that $W$ is a nonnegative, bounded, real valued function defined on $L^p(X)$ which satisfies $W(M) = W(M^*)$.
whenever $M,M^* \in \mathbb{L}^p(X)$ are such that $M_{#} \nu_0 = M^*_{#} \nu_0$. We also assume
\begin{equation}
W \text{ is continuous on } \mathbb{L}^p(X) \text{ endowed with the uniform convergence topology.}
\end{equation}
For $\mu \in \mathcal{P}_p(\mathbb{R}^d)$, $\mathbb{L}^p(\mu)$ is the set of $\xi : \mathbb{R}^d \to \mathbb{R}^d$ that are $\mu$-measurable and such that $||\xi||_p^p := \int_{\mathbb{R}^d} |\xi|^p d\mu < \infty$. We denote by $\mathcal{T}_p \mathcal{P}_p(\mathbb{R}^d)$ the closure of $\{\nabla \varphi : \varphi \in C_c^\infty(\mathbb{R}^d)\}$ in $\mathbb{L}^p(\mu)$. The union of the sets $\{\mu\} \times \mathcal{T}_p \mathcal{P}_p(\mathbb{R}^d)$ is denoted by $\mathcal{T} \mathcal{P}_p(\mathbb{R}^d)$ (cf. e.g. [2]).

We set
\begin{equation}
L(M,N) := \frac{1}{p} ||N||_p^p + W(M), \quad \mathcal{L}(\mu,\xi) := L(M,\xi \circ N), \quad W(\mu) := W(M)
\end{equation}
for $(\mu,\xi) \in \mathcal{T} \mathcal{P}_p(\mathbb{R}^d)$ and $M,N \in \mathbb{L}^p(X)$ such that $M_{#} \nu_0 = \mu$. We assume that $c = 1/p$ and $C$ are such that $L$ satisfies (2.1) and (2.2) and we suppose that (2.23) holds. Note that
\begin{equation}
|W(\mu_1) - W(\mu_2)| \leq \omega(\mathcal{L}^d\{M_1 \neq M_2\})
\end{equation}
for any $\mu_1,\mu_2 \in \mathcal{P}_p(\mathbb{R}^d)$ and any $M_1, M_2 \in \mathbb{L}^p(X)$ such that $\mu_i = M_i_{#} \nu_0, i = 1, 2$.

If $\Omega \subset (0,T)$ is an open set with negligible boundary, we denote $\mathcal{AC}^p(\Omega; \mathcal{P}_p(\mathbb{R}^d))$ the set of $p$-absolutely continuous paths of $\Omega$ into $\mathcal{P}_p(\mathbb{R}^d)$. If $\sigma \in \mathcal{AC}^p(\Omega; \mathcal{P}_p(\mathbb{R}^d))$ and $v$ is the velocity of minimal norm for $\sigma$, we define
\begin{equation}
\mathcal{F}^v(\sigma;\Omega) := \int_\Omega \mathcal{L}(D^{1/\epsilon}_\sigma \sigma_t, v_t \circ D^\epsilon)dt.
\end{equation}
We simply write $\mathcal{F}^v(\sigma, v)$ in place of $\mathcal{F}^v(\sigma;\Omega)$ when $\Omega = (0,T)$.

Remark 5.1. Let $\{\mu^k\}_k \subset \mathcal{P}_p(\mathbb{R}^d)$ be a bounded sequence and $O^k$ be the unique optimal map (optimality being measured against the cost $|\cdot|^p$, unless otherwise specified) that pushes $\nu_0$ forward to $\mu^k$ (cf. e.g. [12]).

(i) Suppose $\{\mu^k\}_k$ converges narrowly to $\mu$. The set $\Gamma_0(\nu_0,\mu)$ which consists of the optimal measures which have $\nu_0$ and $\mu^k$ as marginals, reduces to $\{\text{id} \times O\}$ (cf. e.g. [12]). Here, $O$ is the unique optimal map that pushes $\nu_0$ forward to $\mu$. Proposition 7.1.3 [2] ensures that $\{(\text{id} \times O^k)_{#} \nu_0\}_k$ narrowly converges to $\text{id} \times O$. As a consequence, whenever $1 \leq q < p$, $\{O^k\}_k$ converges to $O$ in $\mathbb{L}^q(X)$.

(ii) Further assume that $\{\mu^k\}_k$ converges to $\mu$ in $\mathcal{P}_p(\mathbb{R}^d)$. Then
\begin{equation}
\limsup_{k \to \infty} ||O^k|| - ||O|| = \limsup_{k \to \infty} |W_p(\delta_0,\mu^k) - W_p(\delta_0,\mu)| \leq \limsup_{k \to \infty} W_p(\mu,\mu^k) = 0.
\end{equation}
As $\{O^k\}_k$ is a bounded subset of $\mathbb{L}^p(X)$ by (i), it converges weakly to $O$ in $\mathbb{L}^p(X)$. Hence, by using (5.3) we conclude that $\{O^k\}_k$ converges to $O$ in $\mathbb{L}^p(X)$.

5.1. Effective Lagrangian on $\mathcal{P}_p(\mathbb{R}^d)$. As in (1.4), we define
\begin{equation}
\mathcal{L}(\mu,\xi) = \liminf_{T \to \infty} \frac{C_{0,T}(\delta_0, (T \xi)_{#} \mu)}{T} = \liminf_{T \to \infty} \frac{C_{0,T}(\delta_0, D^T \circ \xi_{#} \mu)}{T}
\end{equation}
Let $\sigma \in \mathcal{AC}^p(0,T; \mathcal{P}_p(\mathbb{R}^d))$ and let $v$ be a velocity for $\sigma$. Setting $\mathcal{L}(\sigma_t, v_t) := \mathcal{L}(\sigma^t, v^t)$ we have that $v^t$ is a velocity for $\sigma^t$ and $\int_0^T \mathcal{L}(\sigma_t, v_t)dt = \varepsilon \int_0^T \mathcal{L}(\sigma^t, v^t)dt$. We conclude that
\begin{equation}
C_{0,T}(\sigma_0,\sigma_T) = \varepsilon C_{0,T/\epsilon}(D^{1/\epsilon}_\sigma \sigma_0, D^{1/\epsilon}_\sigma \sigma_T).
\end{equation}
In particular,
\begin{equation}
\frac{C_{0,T}(\delta_0,\sigma_T)}{T} = \frac{C_{0,T}(D^{1/\epsilon}_\sigma \sigma_0, D^{T/\epsilon}_\sigma \circ D^{-1/\epsilon}_\sigma \sigma_T)}{\varepsilon / T}.
\end{equation}
Given Lemma 5.3.

Proof: Suppose \( \{\mu^k\}_k \) is a bounded sequence of \( \mathcal{P}_p(\mathbb{R}^d) \) which converges narrowly to \( \mu \) in \( \mathcal{P}_p(\mathbb{R}^d) \). Let \( O, O^k: X \to \mathbb{R}^d \) be such that \( O^k t o \mu^k \) and \( O^k t o \mu \) optimally against \( |\cdot|^p \). Remark 5.1 ensures that \( \{O^k\}_k \) converges to \( O \) in \( L^1(X) \) and so for \( \delta > 0 \) there exist a set \( E \subset X \) such that \( L^d(X \setminus E) \leq \delta \) and up to a subsequence which we do not relabel \( \{O^k\}_k \) converges uniformly to \( O \) on \( E \). Let \( O^k \) be equal to \( O^k \) on \( E \) and \( 0 \) on \( X \setminus E \). We define \( O \) in a similar manner. We exploit (5.2) to obtain

\[
|W(\mu^k) - W(\mu)| \leq |W(O^k) - W(O)| + 2\omega(\delta).
\]

We now use (5.1) by letting \( k \to \infty \), then let \( \delta \) tend to zero we conclude the proof.

\[\Box\]

**Lemma 5.2.** \( W \) is continuous for the narrow convergence on bounded subsets of \( \mathcal{P}_p(\mathbb{R}^d) \).

**Proof:** Suppose \( \{\mu^k\}_k \) is a bounded sequence of \( \mathcal{P}_p(\mathbb{R}^d) \) which converges narrowly to \( \mu \) in \( \mathcal{P}_p(\mathbb{R}^d) \). Let \( O, O^k: X \to \mathbb{R}^d \) be such that \( O^k t o \mu^k \) and \( O^k t o \mu \) optimally against \( |\cdot|^p \). Remark 5.1 ensures that \( \{O^k\}_k \) converges to \( O \) in \( L^1(X) \) and so for \( \delta > 0 \) there exist a set \( E \subset X \) such that \( L^d(X \setminus E) \leq \delta \) and up to a subsequence which we do not relabel \( \{O^k\}_k \) converges uniformly to \( O \) on \( E \). Let \( O^k \) be equal to \( O^k \) on \( E \) and \( 0 \) on \( X \setminus E \). We define \( O \) in a similar manner. We exploit (5.2) to obtain

\[
|W(\mu^k) - W(\mu)| \leq |W(O^k) - W(O)| + 2\omega(\delta).
\]

We now use (5.1) by letting \( k \to \infty \), then let \( \delta \) tend to zero we conclude the proof.

**Lemma 5.3.** Given \( \sigma_a, \sigma_b \in \mathcal{P}_p(\mathbb{R}^d) \), there exists \( \sigma^* \in ACP^p(a,b; \mathcal{P}_p(\mathbb{R}^d)) \) a minimizer in (5.7). Let \( \nu^* \) be the velocity of minimal norm for \( \sigma^* \). Then there exists a set \( N \subset (a,b) \) of zero measure and a map \( \nu: (a,b) \times \mathbb{R}^d \to \mathbb{R}^d \) such that \( \nu_t = \nu^*_t \) for all \( t \in (a,b) \setminus N \) and

\[
t \to \frac{1}{p} \left\| \nu_t \right\|_{L^p(\sigma^*_t)}^p + W(D^{1/p} \sigma^*_t)
\]

is time-independent. In particular,

\[
\sup_{t \in (a,b)} \left\| \nu_t \right\|_{L^p(\sigma^*_t)}^p \leq pC(b-a) \left( 1 + \frac{W_p(\sigma_a, \sigma_b)}{(b-a)^p} \right) + p^2 \sup \|W\|.
\]

**Proof:** Let \( \{\sigma^k\}_k \) be a minimizing sequence in (5.7). Let \( \sigma \) be a geodesic of constant speed connecting \( \sigma_a \) to \( \sigma_b \) and let \( \nu \) be the velocity of minimal norm for it. We assume without loss of generality that

\[
\int_a^b \mathcal{L}(D^{1/p} \sigma_t, \nu_t \circ D^c) \geq \int_a^b \mathcal{L}(D^{1/p} \sigma^k_t, \nu^*_t \circ D^c) dt.
\]

We exploit (5.5), (5.6), the fact that \( \sigma_a^k = \sigma_a \), and use Remark 7.1 to conclude that \( \{\sigma^k\}_k \) is bounded in \( ACP^p(a,b; \mathcal{P}_p(\mathbb{R}^d)) \). In light of the same remark, we may assume without loss of generality the existence of a \( \sigma^* \in ACP^p(a,b; \mathcal{P}_p(\mathbb{R}^d)) \) such that \( \{\sigma^k\}_k \) converges to \( \sigma^* \) in \( \mathcal{P}_q(\mathbb{R}^d) \) for any \( q \in [1,p] \) and every \( t \in [0,T] \). Let \( \nu^* \) be the velocity of minimal norm for \( \sigma^* \). Note that \( \sigma_a^* = \sigma_a \) and \( \sigma_b^* = \sigma_b \). By Lemma 5.2, \( \{W(\nu^*_t)\}_k \) converges to \( W(\sigma^*_t) \) for every \( t \in [0,T] \). As \( \{W(\nu^*_t)\}_k \) is bounded uniformly in \( t \) and \( n \), we use the Lebesgue dominated convergence theorem to conclude that

\[
\lim_{k \to \infty} \int_a^b W(D^{1/p} \sigma^k_t) dt = \int_a^b W(D^{1/p} \sigma^*_t) dt.
\]
If Theorem 5.5.

subsection is to show that $|P|$ of the lemma.

Proof: Let Proposition 5.4. Then one can reproduce verbatim the proof of Proposition 3 [13] given in the case $p = 2$ to infer

$$\lim_{k \to +\infty} \inf \int_a^b \|v_k\|_{L^p(\sigma^*_t)}^p dt \geq \int_a^b \|v_t^*_i\|_{L^p(\sigma^*_t)}^p dt,$$

which is the last ingredient needed to conclude that $\sigma^*$ is a minimizer in (5.7).

We have the distributional derivative of $t \mapsto 1/p'\|v_t\|_{L^p(\sigma^*_t)}^p + W(D^{1/p'}\sigma^*_t)$ is null (cf. Proposition 3.11 [14] for the case $p = 2$). This proves that there exists a set $N \subset (0, T)$ of zero measure and a map $v : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$ such that $v_t = v_t^*$ for all $t \in (0, T) \setminus N$ and $t \to 1/p'\|v_t\|_{L^p(\sigma^*_t)}^p + W(D^{1/p'}\sigma^*_t)$ is time-independent. By (5.6), the set of $t$ such that

$$\frac{1}{p'}\|v_t\|_{L^p(\sigma^*_t)}^p - W(D^{1/p'}\sigma^*_t) \leq C(b-a)\left(1 + \frac{W_p^p(\sigma_a, \sigma_b)}{(b-a)^p}\right)$$

is of positive measure. We use that $1/p'\|v_t\|_{L^p(\sigma^*_t)}^p + W(D^{1/p'}\sigma^*_t)$ is time-independent to conclude the proof of the lemma.

\[\square\]

5.2. $P_p(\mathbb{R}^d)$ as a quotient space of $L^p(X)$.

Proposition 5.4. Let $M \in L^p(X)$ and fix $\delta > 0$. Then

(i) There exist $M \in L^p(X)$ and a partition of $X$ into parallel cubes $\{X_i\}_{i=1}^k$ of same size such that $\|M - M\| \leq \delta$, $|M| \leq |M| + 1$, $M$ is a constant value $m_i$ on each $X_i$, $m_i \in \mathbb{Q}$ and $m_i \neq m_j$ whenever $i \neq j$.

(ii) There exists $M \in L^p(X)$ such that $|M| \leq |M| + 2$, $\|M - M\| \leq \delta$, $M$ is invertible, both $M$ and $M^{-1}$ are Borel maps. Furthermore, $L^d(M^{-1}(A)) = 0$ whenever $A \subset \mathbb{R}^d$ and $L^d(A) = 0$.

Proof: (i) Clearly, there exists $M^* \in L^p(X)$ and a partition of $X$ into parallel cubes $\{X_i\}_{i=1}^k$ of same size such that $|M| \leq |M| + 1/2$, $\|M^* - M\| \leq \delta/2$, $M^*$ is a constant value $m^*_i$ on each $X_i$ and $\{m^*_i\}_{i=1}^k \subset \mathbb{Q}$. Next, choose a set $\{m_i\}_{i=1}^k \subset \mathbb{Q}$ of cardinality $k$ such that $2|m_i - m_j| \leq \min\{1, \delta\}$. Then $M = \sum_{i=1}^k m_i \chi_{X_i}$ satisfies the required property.

(ii) In light of (i), we may assume without loss of generality that there exists a partition of $X$ into parallel cubes $\{X_i\}_{i=1}^k$ of same size such that $M$ is a constant value $m_i$ on each $X_i$, $m_i \in \mathbb{Q}$ and $m_i \neq m_j$ whenever $i \neq j$. Set $\tilde{M} z = M z + z/a$ where $a > 0$ is such that

$$\min_{i \neq j} |m_i - m_j| > \sqrt{d}, \quad 2\sqrt{d} < a\delta, \quad a > \sqrt{d}.$$

The first inequality above ensures that $\tilde{M}$ is one-to-one, the second one gives that $\|\tilde{M} - M\| \leq \delta$, while the third one ensures that $|\tilde{M}| \leq |M| + 1$. Clearly, both $M$ and $M^{-1}$ are Borel maps and $L^d(M^{-1}(A)) = 0$ whenever $A \subset \mathbb{R}^d$ and $L^d(A) = 0$.

The push forward operator $M \mapsto M\#\nu_0$ defines an equivalence relation on $L^p(X)$. The goal of this subsection is to show that $M\#\nu_0 = M\#\nu_0$ if and only if $\inf G \|M - M \circ G\| = 0$ where, $G$ is the non commutative group which consists of maps $G : X \to X$ such that $G$ is one-to-one $L^d$-almost everywhere onto $X$, and $G, \ G^{-1}$ are Borel maps that push $\nu_0$ forward to itself.

Theorem 5.5. If $M, \tilde{M} \in L^p(X)$, then the following are equivalent:

(i) $M\#\nu_0 = \tilde{M}\#\nu_0$;

(ii) $(\exists)\{G^n\}_n \subset G$ such that $\lim_{n \to \infty} \|M - \tilde{M} \circ G^n\| = 0.$
Proof: We will prove that (i) implies (ii). The reverse implication is easy. Set \( \mu = M \# \nu_0 \) and suppose that (i) holds. Let \( O \) be the unique \( p \)-optimal Borel map that pushes \( \nu_0 \) forward to \( \mu \) (i.e. optimality is measured against the cost function \( | \cdot |^p \)). We may assume without loss of generality that \( M = O \). Let \( \{ M^n \}_n \subset \mathcal{L}^\infty(X) \) be a sequence of Borel maps converging to \( \tilde{M} \) in \( \mathbf{L}^p(X) \) such that \( M^n \) is invertible, both \( M^n \) and its inverse are Borel maps and \( \mathcal{L}^d \left( \left( M^n \right)^{-1}(A) \right) = 0 \) whenever \( A \subset \mathbb{R}^d \) is a Borel measurable set such that \( \mathcal{L}^d(A) = 0 \).

Such a sequence of maps exists by Proposition 5.4.

Let \( \{ O^n \}_n \subset C(\mathbb{R}^d) \) be a sequence of maps such that \( O^n \# \nu_0 = M^n \# \nu_0 := \mu^n \) and \( O^n \) is \( p \)-optimal. Such a sequence exists by the mass transportation theory (cf. e.g. [12]) and \( O^n \) is essentially invertible. Set

\[
F^n := (O^n)^{-1} \circ M^n, \quad G^n := (M^n)^{-1} \circ O^n.
\]

Note that \( F^n \) and \( G^n \) are Borel maps as compositions of Borel maps. They preserve the Lebesgue measure and are both one-to-one almost everywhere from \( X \) onto \( X \). In fact, \( G^n \) is essentially the inverse of \( F^n \). The convergence of \( \{ M^n \}_n \) to \( \tilde{M} \) in \( \mathbf{L}^p(X) \) ensures that of \( \{ \mu^n \}_n \) to \( \mu \) in \( \mathcal{P}_p(\mathbb{R}^d) \) and so, by Remark 5.1. \( \{ O^n \}_n \) converges to \( O \) in \( \mathbf{L}^p(X) \). Note

\[
\tilde{M} \circ G^n - M = \tilde{M} \circ G^n - M^n \circ G^n + O^n - O.
\]

Using the triangular inequality and the fact that \( G^n \) preserves Lebesgue measure we obtain

\[
\limsup_{n \to \infty} \| M - \tilde{M} \circ G^n \| \leq \limsup_{n \to \infty} \| M^n - \tilde{M} \| + \| O^n - O \| = 0.
\]

\[\square\]

Remark 5.6. Let \( W : \mathbf{L}^p(X) \to \mathbb{R} \) be a continuous map. By Proposition 5.5, if \( W(M) = W(M \circ G) \) for all \( M \in \mathbf{L}^p(X) \) and all \( G \in \mathcal{G} \), then \( W(M) = W(\tilde{M}) \) for all \( M, \tilde{M} \in \mathbf{L}^p(X) \) satisfying \( M \# \nu_0 = \tilde{M} \# \nu_0 \). The converse of this statement clearly holds even if \( W \) is not continuous.

5.3. Homogenization on \( \mathcal{P}_p(\mathbb{R}^d) \). Lemma 3.5 suggests that in higher dimensions we consider a special subset of \( AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d)) \) on which we introduce a special topology.

Definition 5.7. Let \( \sigma \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d)) \) and suppose \( v \) is the velocity of minimal norm for \( \sigma \). Suppose \( \phi \in \mathcal{H} \) is such that

\[
\begin{align*}
(i) & \quad \phi_t \# \nu_0 = \sigma_t \text{ for every } t \in [0, T]; \\
(ii) & \quad \dot{\phi}_t = v_t \circ \phi_t \text{ for a.e. } t \in (0, T).
\end{align*}
\]

We say that \( \phi \) is a flow associated to \( (\sigma, v) \). We denote by \( S_\sigma(\mathbb{R}^d) \) the set of \( \sigma \) such that \( \sigma \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d)) \), \( v \) is the velocity of minimal norm for \( \sigma \), and there is a flow \( \phi \) associated to \( (\sigma, v) \).

Definition 5.8. Let \( \sigma \in S_\sigma(\mathbb{R}^d) \) and suppose \( v \) is the velocity of minimal norm for \( \sigma \). Let \( \{ \sigma^n \}_n \subset S_\sigma(\mathbb{R}^d) \) and suppose \( v^n \) is the velocity of minimal norm for \( \sigma^n \). We say that \( \{ \sigma^n \}_n \) \( \tau \)-converges to \( \sigma \) if there exists a flow \( \phi^n \) associated to \( (\sigma^n, v^n) \) and a flow \( \phi \) associated to \( (\sigma, v) \) such that \( \{ \phi^n \}_n \) \( \tau \)-converges to \( \phi \) in \( AC^p(0, T; \mathbf{L}^p(X)) \).

Definition 5.9. (i) We say that a subset \( \mathcal{A} \) of \( AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d)) \) is bounded if

\[
\sup_{\mathcal{A}} \int_0^T \left( W^p_p(\sigma, \delta_0) + |\sigma'|^p(t) \right) dt < \infty.
\]

(ii) We say that a bounded sequence \( \{ \sigma^n \}_n \) \( \tau_w \)-converges to \( \sigma \) in \( AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d)) \) if

\[
\lim_{n \to \infty} \int_0^T W_1(\sigma^n, \sigma_t) dt = 0.
\]
Corollary 5.10. Suppose \( \{\sigma^n\}_n \subset AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d)) \) \( \tau_w \)-converges to \( (\sigma) \) in \( AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d)) \). Let \( O^p_\eta \) be the \( p \)-optimal map that pushes \( \nu_0 \) forward to \( \sigma^k \). Similarly, let \( O_t \) be the \( p \)-optimal map that pushes \( \nu_0 \) forward to \( \sigma_t \). If \( q \in [1,p) \), then \( \{O^p_\eta\}_n \) converges to \( O \) in \( L^q(X) \). As a consequence, for each \( t \in [0,T] \), \( \{\sigma^n_t\}_n \) converges to \( \sigma_t \) in \( \mathcal{P}_q(\mathbb{R}^d) \).

Proof: By Remark 7.1, \( \sigma^n \) is \( 1/p' \)-Hölder continuous uniformly in \( n \) and so we exploit the fact that \( \lim_{n \to \infty} \int_0^T W_1(\sigma^n_t, \sigma_t)dt = 0 \) to conclude that every subsequence of \( \{\sigma^n\}_n \) admits a subsequence \( \{\sigma^{n_k}\}_k \) such that not only does \( \sigma^{n_k} \) converge to \( \sigma_t \) in \( \mathcal{P}_1(\mathbb{R}^d) \), but it also converges in \( \mathcal{P}_q(\mathbb{R}^d) \) for each \( t \in [0,T] \). By Remark 3.3, \( \{O^{n_k}\}_k \) converges to \( O \) in \( L^q(X) \). The limit being independent of the subsequence we started with, we conclude that, in fact, for the whole sequence we have that \( \{O^p_\eta\}_n \) converges to \( O \) in \( L^q(X) \). We use that \( W_q(\sigma^n_t, \sigma_t) \leq \|O^p_t - O_t\|_{\mathcal{L}^q(X)} \) to conclude that for each \( t \in [0,T] \), \( \{\sigma^n_t\}_n \) converges to \( \sigma_t \) in \( \mathcal{P}_q(\mathbb{R}^d) \). \( \square \)

Remark 5.11. Every \( \tau \)-convergent sequence in \( \mathcal{S}_p(\mathbb{R}^d) \) is \( \tau_w \)-convergent in \( AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d)) \).

Proof: Assume \( \{\sigma^n\}_n \tau \)-converges to \( \sigma \) in \( \mathcal{S}_p(\mathbb{R}^d) \). Let \( v^n \) (resp. \( v \)) be the velocity of minimal norm for \( \sigma^n \) (resp. \( \sigma \)). Let \( \phi^n \) (resp. \( \phi \)) be a flow associated to \( \sigma^n \) (resp. \( \sigma \)) such that (2.15–2.17) hold. Let \( Y \in X \) be an open set. Then

\[
\int_0^T W_1(\sigma^n_t, \sigma_t)dt \leq \mathcal{L}^d(Y)^{1/p} \|\phi^n - \phi\|_{\mathcal{L}^p((0,T) \times Y)} + \mathcal{L}^d(X \setminus Y)^{1/p} \sup_n \|\phi^n - \phi\|_{\mathcal{L}^p((0,T) \times X)}.
\]

We let \( n \) tend to \( \infty \), use (2.13) and then let \( Y \) tend to \( X \) to obtain that the left hand side tends to zero. As \( W_p(\sigma^n_t, \sigma_t) \leq \|\phi^n - \phi\|_p \), we conclude that \( |(\sigma^n)'(t)| \leq |\phi^n(t)| \) for almost every \( t \in (0,T) \). Thus,

\[
\int_0^T W_p(\sigma^n_t, \sigma_t, \delta_0)dt = \|\phi^n\|_{\mathcal{L}^p((0,T) \times X)}^p + \int_0^T |(\sigma^n)'(t)|^p dt \leq \|\phi^n\|_{\mathcal{L}^p((0,T) \times X)}^p.
\]

Hence, (2.14 – 2.15) hold. \( \square \)

5.4. A topology stronger than \( \tau_w \) yielding the same \( \Gamma \)-limit. Recall that \( \omega \) is the continuous, monotone nonincreasing function introduced in (5.2) such that \( \omega(0) = 0 \).

Let \( K \equiv K^m \subset C^1(\mathbb{R}^d;\mathbb{R}^d) \) be a 1-Lipschitz function such that \( |K(x)| \leq m \) for all \( x \in \mathbb{R}^d \), \( K(x) = 0 \) for \( |x| \geq m + 2 \) and \( K(x) = x \) for \( |x| \leq m \). As \( K \) is 1-Lipschitz, so is the operator \( \mu \to K_{\#}\mu \) of \( \mathcal{P}_q(\mathbb{R}^d) \) into itself, for any \( q \in [1,\infty) \). If \( t,s \in [0,T] \) then

\[
W_q(K_{\#}\sigma_t, K_{\#}\sigma_s) \leq W_q(\sigma_t, \sigma_s), \quad W_q(K_{\#}\mu, \mu) \leq 2 \left( \int_{|x| \geq m} |x|^q \mu(dx) \right)^{1/q}.
\]

Lemma 5.12. If \( \sigma \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d)) \), then for any map \( \phi \in \mathcal{L}^p((0,T) \times X) \) such that \( \phi_{t\#}\nu_0 = \sigma_t \) we have

\[
F^\prime(\sigma) \geq F^\prime(K_{\#}\sigma) - \int_0^T \omega\left( \mathcal{L}^d(|\phi_t| \geq m) \right) dt \geq F^\prime(K_{\#}\sigma) - \int_0^T \omega\left( \frac{W_1(\sigma_t, \delta_0)}{m} \right) dt.
\]

Proof: Note that the first inequality in the lemma implies the second one. Our task reduces then to prove the first inequality. By (5.10) \( \sigma \to K_{\#}\sigma \) is 1-Lipschitz and maps \( AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d)) \) into itself. Thus, \( |(K_{\#}\sigma)'(t)| \) is at most \( |\sigma'(t)| \) almost everywhere on \( (0,T) \). Fix \( a > 0 \) and \( M \in \mathcal{L}^p(X) \) such that \( M_{\#}\nu_0 = \mu \in \mathcal{P}_p(\mathbb{R}^d) \). We have

\[
|W(D^a_{\#}\mu) - W((D^a \circ K)_{\#}\mu)| = |W(D^a \circ M) - W(D^a \circ K \circ M)| \leq \omega\left( \mathcal{L}^d\{D^a \circ M \neq D^a \circ K \circ M\} \right) \leq \omega\left( \mathcal{L}^d\{|M| \geq m\} \right).
\]

This is the last ingredient to conclude the proof of the lemma. \( \square \)
Corollary 5.13. Suppose $\sigma \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$ and $\sigma_t$ is supported in the ball of radius $R < m$ centered at the origin for every $t \in [0,T]$. Then there exists $\{\sigma^n\}_n$ $\tau_w$-converging to $\sigma$ such that for every $t \in [0,T]$ and every $n \in \mathbb{N}$, $\sigma^n_t$ is supported in the ball of radius $m+2$ centered at the origin, and $F_w(\sigma) = \lim_{n \to \infty} F_w^n(\sigma^n)$. As a consequence, $\{\sigma^n_t\}_n$ converges to $\sigma_t$ in $\mathcal{P}_p(\mathbb{R}^d)$ for every $t \in [0,T]$.

**Proof:** Let $\{\tilde{\sigma}^n\}_n$ be a sequence $\tau_w$-converging to $\sigma$ and such that $F(\sigma) = \lim_{n \to \infty} F_w^n(\tilde{\sigma}^n)$. Set $\sigma^n := K_\# \tilde{\sigma}^n$. By (5.10) and the fact that $|\sigma^n(t)| \leq |\tilde{\sigma}^n(t)|$ almost everywhere on $(0,T)$, $\{\sigma^n\}_n$ $\tau_w$-converges to $K_\# \sigma = \sigma$. Let $O^n_t$ be the $p$-optimal map that pushes $\tilde{\sigma}^n_t$ forward to $\sigma^n_t$ and, similarly, let $O_t$ be the $p$-optimal map that pushes $\tilde{\sigma}_t$ forward to $\sigma_t$. Corollary 5.10 gives that $\{O^n_t\}_n$ converges to $O_t$ in $L^1(X)$ and so, since $|O_t| \leq R < m$, we conclude that $\liminf_{n \to \infty} L^d(O^n_t \geq m) = 0$. We use Lemma 5.12 to conclude that

$$F(\sigma) = \lim_{n \to \infty} F_w^n(\tilde{\sigma}^n) \geq \liminf_{n \to \infty} \left\{ F_w^n(\sigma^n) - \int_0^T \omega \left( L^d(O^n_t \geq m) \right) dt \right\} \geq F(K_\# \sigma) = F(\sigma).$$

By Corollary 5.10 $\{\sigma^n_t\}_n$ converges to $\sigma_t$ in the $W_1$-metric and so, $\{\sigma^n_t\}_n$ converges to $\sigma_t$ in the $W_p$-metric. Note that $\sigma^n_t$ is supported by the ball of radius $m+2$ centered at the origin. Hence, $\{\sigma^n_t\}_n$ converges to $\sigma_t$ in the $W_p$-metric. □

Corollary 5.14. Let $\sigma \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$. The following hold:

(i) There exists a sequence $\{\sigma^m\}_m$ $\tau_w$-converging to $\sigma$ such that $F_w(\sigma) = \lim_{m \to \infty} F_w(\sigma^m)$ and for each $m \in \mathbb{N}$ there exists $r^m > 0$ such that $\sigma^m_t$ is contained in the ball of radius $r^m$, centered at the origin.

(ii) There exists a sequence $\{\sigma^n\}_n$ $\tau_w$-converging to $\sigma$ such that $F_w(\sigma) = \lim_{n \to \infty} F_w^n(\sigma^n)$ and $\{\sigma^n_t\}_n$ converges to $\sigma_t$ in $\mathcal{P}_p(\mathbb{R}^d)$ for every $t \in [0,T]$.

**Proof:** (i) We now denote $K$ by $K^m$ to display its dependence on $m$. Let $\sigma \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$, set $\sigma^n_t = K^m_\# \sigma_t$ and let $\delta > 0$ be arbitrary. By the fact that $\sigma_0 \in \mathcal{P}_p(\mathbb{R}^d)$, Remark 7.1 yields $\sup_t W_p(\sigma_t, \delta_0) < \infty$.

Thus, for $m$ large enough $\omega \left( W_{\mathcal{P}}(\sigma_t, \delta_0) \right) < \delta$. We exploit Lemma 5.12 to obtain that $F_w(\sigma) \geq F_w(\sigma^m) - \delta$. As $F_w$ is $\tau_w$-lower semicontinuous and $\{\sigma^m\}_m$ $\tau_w$-converges to $\sigma$, we conclude that

$$-\delta + \limsup_{m \to +\infty} F_w(\sigma^m) \leq F_w(\sigma) \leq \liminf_{m \to +\infty} F_w(\sigma^m).$$

As $\delta > 0$ is arbitrary, we conclude the proof of (i).

(ii) For each $k \in \mathbb{N}$, choose $k \leq m_k \in \mathbb{N}$ large enough such that $F_w(\sigma) \geq F_w(K^{m_k}_\# \sigma) - 1/k$ and set $\sigma^k_t := K^{m_k}_\# \sigma_t$. By Corollary 5.13, we can choose a sequence $\{\sigma^{k,n}\}_n$ that $\tau_w$-converges to $\sigma^k$ and such that

$$F_w(\sigma^k) \geq F_w^n(\sigma^{k,n}) \text{ and } \lim_{n \to \infty} W_p(\sigma^{k,n}_t, \sigma^k_t) = 0$$

for each $t \in [0,T]$. Thanks to Egoroff’s Theorem, there exists a decreasing sequence of Lebesgue measurable sets $N_1 \supset N_2 \supset \cdots \supset N_k \supset \cdots$ such that $\mathcal{L}^d(N_k) < 1/k$ and there exists $n_k$ such that

$$F_w(\sigma^k) \geq F_w^n(\sigma^{k,n}) - \frac{1}{k} \text{ and } W_p(\sigma^{k,n}_t, \sigma^k_t) < \frac{1}{k}$$

for all $n \geq n_k$ and all $t \not\in N_k$. We exploit (5.10) and the triangle inequality to obtain that

$$W_p(\sigma^{k,n}_t, \sigma_t) \leq W_p(\sigma^{k,n}_t, \sigma^k_t) + W_p(\sigma^k_t, \sigma_t) \leq \frac{1}{k} + 2 \left( \int_{|x| > k} |x|^p \sigma_t(dx) \right)^{\frac{1}{p}}$$

for all $t \not\in N := \cap_{k=1}^\infty N_k$. Hence, $\lim_{k \to \infty} W_p(\sigma^{k,n}_t, \sigma_t) = 0$ for all $t \not\in N$. As $W \geq 0$, we have that

$$\int_0^T |(\sigma^{k,n})'(t)| dt \leq pF_w^n(\sigma^{k,n}) \leq pF_w(\sigma) + p.$$
We use this uniform bound in $n$ and $k$ and Remark 7.1 to conclude that the sequence $\{\sigma_{k,n}^t\}_k$ is bounded in $AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$ and $\{\sigma_{k,n}^t\}_k$ is uniformly Hölder continuous. Hence,

$$\lim_{k \to \infty} W_p(\sigma_{k,n}^t,\sigma_t) = 0 \text{ for all } t \in [0,T].$$

Using that $W_1 \leq W_p$, we conclude that $\{\sigma_{k,n}^t\}_k$ $\tau_w$-converges to $\sigma$. As a consequence,

$$\mathcal{F}_w(\sigma) \leq \lim \inf_{k \to \infty} \mathcal{F}_{\tau_n}(\sigma_{k,n}^t).$$

But, we also have

$$\mathcal{F}_w(\sigma) \geq \lim \sup_{k \to \infty} \mathcal{F}_{\tau_n}(\sigma_{k,n}^t),$$

which concludes the proof of the corollary. \qed

**Remark 5.15.** Let $\tau_w$ be the topology obtained by replacing $\lim_{n \to \infty} \int_0^T W_1(\sigma_n^t,\sigma_t)dt = 0$ by the stronger condition $\lim_{n \to \infty} \int_0^T W_p(\sigma_n^t,\sigma_t)dt = 0$ in definition 5.9. We have proved above that the $\Gamma(\tau_w)$-limit of $\mathcal{F}_\tau$ is still $\mathcal{F}_w$.

5.5. **Representation of the effective Lagrangians on $\mathcal{P}_p(\mathbb{R}^d)$ in terms of that on $L^p(X)$**.

**Remark 5.16.** Let $\sigma \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$ and let $v$ be a velocity for $\sigma$. We extend $\sigma$ outside $[0,T]$ by setting $\sigma_t = \sigma_0$ if $t < 0$ and $\sigma_t = \sigma_T$ if $t > T$. Similarly, we extend $v$ by setting $v_t = 0$ if $t < 0$ or $t > T$. Note that for any $a < b$, the extension (still denoted by $\sigma$) belongs to $AC^p(a,b;\mathcal{P}_p(\mathbb{R}^d))$ and the extension of $v$ remains a velocity for $\sigma$. If $v$ is the velocity of minimal norm, then so is its extension.

(i) One can use a rescaled family of spatial mollifiers $\rho^\epsilon > 0$ and define

$$\sigma_i^\epsilon = \rho^\epsilon * \sigma_i \text{ and } \sigma_i^\epsilon v_i^\epsilon = \rho^\epsilon * (\sigma_i v_i).$$

We have $\|v_i^\epsilon\|_{\sigma_i^\epsilon} \leq \|v_i\|_{\sigma_i}$, (cf. e.g. Lemma 8.1.9 [2]). Then one uses a rescaled family of time mollifiers $\nu^\delta$ supported in $[-\delta,\delta]$ and defines

$$\sigma_i^\epsilon,\delta = \nu^\delta * \sigma_i^\epsilon, \quad \sigma_i^\epsilon,\delta v_i^\epsilon = \nu^\delta * (\sigma_i^\epsilon v_i^\epsilon).$$

We have $\|v_i^\epsilon,\delta\|_{\sigma_i^\epsilon,\delta} \leq \|v_i^\epsilon\|_{\sigma_i^\epsilon}$ (cf. e.g. [11] subsection 5.3). We have that

$$\sigma_i^\epsilon \to \sigma_i \text{ in } \mathcal{P}_p(\mathbb{R}^d), \quad \sigma^\epsilon v \to \sigma v \text{ narrowly and } \|v_i^\epsilon\|_{\sigma_i^\epsilon} \to \|v_i\|_{\sigma_i},$$

as $\epsilon \to 0$. Similarly,

$$\sigma_i^\epsilon,\delta \to \sigma_i \text{ in } \mathcal{P}_p(\mathbb{R}^d), \quad \sigma^\epsilon,\delta v \to \sigma v \text{ narrowly and } \|v_i^\epsilon,\delta\|_{\sigma_i^\epsilon,\delta} \to \|v_i^\epsilon\|_{\sigma_i^\epsilon}$$

as $\delta \to 0$. Furthermore,

$$\sigma^\epsilon,\delta \in C^\infty([0,T] \times \mathbb{R}^d) \quad \text{and} \quad v^\epsilon,\delta \in C^\infty([0,T] \times \mathbb{R}^d;\mathbb{R}^d).$$

(ii) In conclusion, for each integer $k \geq 1$ there exist $\{\sigma_k\}_k \subset AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$ and velocities $v_k$ for $\sigma_k$ such that $\sigma_k \in C^\infty([0,T] \times \mathbb{R}^d)$ and $v_k \in C^\infty([0,T] \times \mathbb{R}^d;\mathbb{R}^d)$. Furthermore $\sigma_k^T \to \sigma_T$ in $\mathcal{P}_p(\mathbb{R}^d)$ for each $t \in [0,T]$, $\sigma_k v_k \to \sigma v$ narrowly and $\|v_i^\epsilon\|_{\sigma_i^\epsilon,\delta} \to \|v_i\|_{\sigma_i}$, as $k \to \infty$. As $W$ is bounded, the Lebesgue dominated convergence theorem yields

$$\lim_{k \to \infty} \int_0^T \mathcal{L}(\sigma_k^T, v_k^T)dt = \int_0^T \mathcal{L}(\sigma_T, v_T)dt.$$

(iii) By Proposition 8.1.8 [2], there exists $\{S^k\}_k \subset W^{1,p}(0,T;L^p(\sigma_0))$ such that

$$\dot{S}^k_t(x) = v_k^T \circ S^k_t(x) \text{ and } S_0^k(x) = x \text{ for a.e. } x \in \mathbb{R}^d \text{ and } t \in [0,T].$$

Since $\sigma_0^T > 0$ and

$$\int_0^T \left( \sup_B |v_k^T| + \text{Lip}(v_k^T, B) \right)dt < \infty$$
for every ball $B \subseteq \mathbb{R}^d$, we conclude that, in fact, (5.11) holds for every $x$. Furthermore, $S_t^k$ is invertible and $S_t^k \nu_0 = \sigma_t^k$.  

**Theorem 5.17.** If $\mu \in \mathcal{P}_p(\mathbb{R}^d)$, $V \in \mathbf{L}^p(\mu)$, $M \in \mathbf{L}^p(X)$ is any that pushes the Lebesgue measure restricted to $X$ forward to $V_{\#} \mu := \nu$ then $\bar{L}(M) = \bar{L}(\mu, V)$.

**Proof:** 1. Let $(\sigma, v)$ be such that $v$ is a velocity for $\sigma \in \mathcal{P}(\delta_0, D^T_# \nu)$, where we recall the definition of $D^T$: $D^T x = T x$. For $k \geq 1$ integer we choose $\sigma^k$ and $v^k$ as in Remark 5.16. We set $\nu^k_t := D^T_#^{-1} \sigma_t^k$ and $\nu_t := D^T_#^{-1} \sigma_t$ so that

$$
\lim_{k \to \infty} W_p^p(\nu^k_t, \nu_t) = 0.
$$

Fix $\delta > 0$ and $k$ such that

$$
\int_0^T \mathcal{L}(\sigma^k_t, v^k_t) dt \geq \int_0^T \mathcal{L}(\sigma^k_t, v_t) dt - \delta.
$$

Let $S^k$ be as in Remark 5.16. Let $O^{k}_{T} : X \to \mathbb{R}^d$ be the unique $|\cdot|^p$-optimal map that pushes the Lebesgue measure restricted to $X$ forward to $\sigma^k_T$ (cf. e.g. [12] Theorem 3.7) and, similarly, let $O^k : X \to \mathbb{R}^d$ be the unique $|\cdot|^p$-optimal map that pushes the Lebesgue measure restricted to $X$ forward to $\nu$. Set

$$
M^k_t := S^k_t \circ (S^k_T)^{-1} \circ D^T \circ O^k_T,
$$

so that $M^k_t$ pushes the Lebesgue measure restricted to $X$ forward to $\sigma^k$.

By (5.11)

$$
M^k_t = v^k_t \circ M^k_t
$$

for $t \in [0, T]$ and $\{M^k\}_k \subseteq \mathcal{H}$. We have

$$
\int_0^T \mathcal{L}(\nu^k_t, v^k_t) dt = \int_0^T \mathcal{L}(\sigma^k_t, v^k_t) dt \geq C_{0,T}(M^0_0, O^{k}_T).
$$

By (2.6),

$$
C_{0,T}(M^0_0, TO) \leq C_{0,1}(0, M^0_0) + C_{1,T-2}(M^0_0, (T - 3)O^{k}_{T-3}) + C_{T-2,T-1}((T - 3)O^{k}_{T-3}, (T - 3)O^k) + C_{T-1,T}((T - 3)O^k, TO) \leq C_{0,T-3}(M^k_0, (T - 3)O^{k}_{T-3}) + C \left( 3 + \|M^k_0\|^p + (T - 3)^p\|O^{k}_{T-3} - O^k\|^p \right),
$$

where we have used (2.1). By Remark 5.1 and (5.12), $\{O^{k}_T\}_k$ converges strongly to $O$ in $\mathbf{L}^p(X)$.

We use Remark 5.16 and the fact that $M^k_0$ pushes the Lebesgue measure restricted to $X$ forward to $\sigma^k_0$ to conclude that

$$
\|M^k_0\| = W_p^p(\sigma^k_0, \delta_0) = 0(1/k)
$$

and so, $\{M^k_0\}_k$ converges to 0 in $\mathbf{L}^p(X)$. Similarly, Letting first $k$ tend to $\infty$ in the previous inequality we obtain

$$
C_{0,T}(M^0_0, TO) \leq \liminf_{k \to \infty} C_{0,T-3}(M^k_0, (T - 3)O^{k}_{T-3}) + 3^pC\|O\|^p.
$$

We combine this, together with (5.13) and (5.14) and use that $\sigma$ is an arbitrary path starting at 0 and ending at $D^T_# \nu$, $\delta > 0$ is arbitrary to conclude that

$$
C_{0,T}(\delta_0, D^T_# \nu) \geq C_{0,T+3}(M^0_0, (T + 3)O^k - 3^pC\|O\|^p).
$$

Dividing both sides of (5.15) by $T$ and letting $T$ tend to $\infty$ we conclude that $\mathcal{L}(\mu, v) \geq \bar{L}(O)$. As $\bar{L}$ is locally Lipschitz and so, it is continuous, we use Theorem 5.5 and the fact that $M_# \nu_0 = O_# \nu_0$ to conclude that $\mathcal{L}(\mu, V) \geq \bar{L}(M)$. 

Conversely, let $\phi \in \mathcal{H}$ be such that $\phi_0 \equiv 0$ and $\phi_T = TM$. We exploit Corollary 2.3 and the fact that $W$ is continuous to obtain $\phi^k \in \mathcal{H}$ and a partition of $X$ into parallel cubes $\{X_i\}_{i=1}^k$ of same size such that $\|\phi^k - \phi_t\| \leq 1/k$, $\phi_t$ is constant on each cube for $t \in [0, T]$ and

\[
\int_0^T L(\phi_t, \dot{\phi}_t)dt \geq \int_0^T L(\phi^k_t, \dot{\phi}^k_t)dt - \frac{1}{k}.
\]

Set $\sigma^{k,T}_t$ be the push forward by $\phi^k_t$ of the Lebesgue measure restricted to $X$ and let $\sigma^T_t$ be the push forward by $\phi_t$ so that

\[
W_p(\sigma^{k,T}_t, \sigma^T_t) \leq \|\phi^k_t - \phi_t\|^p \leq 1/k^p.
\]

We have added the subscript $T$ to emphasize the $T$-dependence of $\sigma^{k,T} : [0, T] \to \mathcal{P}_p(\mathbb{R}^d)$. By Lemma 7.4, there exists a velocity $w^k$ for $\sigma^{k,T}$ such that $\phi^k_t = w^k_t \circ \phi^k_t$ for almost every $t \in (0, T)$. We use (5.16) and the fact that

\[
\int_0^T dt \int_X \|\phi^k_t\|^p dz = \int_0^T dt \int_X \|w^k_t\|^p d\sigma^{k,T}_t
\]

for all $t \in [0, T]$ to conclude that

\[
\int_0^T L(\phi_t, \dot{\phi}_t)dt \geq C_{0,T}(\sigma^{k,T}_0, \sigma^T_T) - \frac{1}{k}.
\]

But, as above,

\[
C_{0,T}(\delta_0, \sigma^T_T) \leq C_{0,1}(\delta_0, \sigma^0_T) + C_{0,T-3}(\sigma^{k,T-3}_0, \sigma^{k,T-3}_T) + C_{0,1}(\sigma^{k,T-3}_T, \sigma^{T-3}_T) + C_{0,1}(\sigma^{T-3}_T, \sigma^T_T)
\]

By (2.1),

\[
C_{0,1}(\delta_0, \sigma^0_T) \leq C(1 + W_p(\delta_0, \sigma^0_T)) = C(1 + \|M^0\|^p).
\]

Similarly,

\[
C_{0,1}(\sigma^{k,T-3}_T, \sigma^{T-3}_T) \leq C\left(1 + \frac{1}{k^p}\right),
\]

\[
C_{0,1}(\sigma^{T-3}_T, \sigma^T_T) \leq C(1 + 3^p\|M\|^p).
\]

Letting $k$ tend to infinity in (5.18), we conclude that

\[
C_{0,T}(\delta_0, \sigma^T_T) \leq \lim_{k \to \infty} C_{0,T-3}(\sigma^{k,T-3}_0, \sigma^{k,T-3}_T) + C_{0,1}(\sigma^{T-3}_T, \sigma^T_T).
\]

As $\phi$ is an arbitrary path such that $\phi_0 \equiv 0$ and $\phi_T = TM$, (5.17) and (5.19) imply

\[
C_{0,T}(0, TM) \geq C_{0,T}(\delta_0, (D^T \circ V) \# \nu_0).
\]

Dividing both sides of the inequality by $T$ and letting $T$ tend to infinity we have $\tilde{L}(M) \geq \tilde{L}(\mu, V)$. 

\[\square\]

**Corollary 5.18.** Let $\sigma \in \mathcal{S}_p(\mathbb{R}^d)$ and let $v$ be the velocity of minimal norm for $\sigma$. Whenever $\phi \in AC^p(0, T; L^p(X))$ and $\phi_t \# \nu_0 = \sigma_t$ for all $t \in [0, T]$, we have

\[
\dot{L}(v_t \circ \phi_t) = \tilde{L}(\sigma_t, v_t) \quad \forall t \in [0, T].
\]

**Proof:** Let $\phi \in AC^p(0, T; L^p(X))$ be such that $\phi_t \# \nu_0 = \sigma_t$ for all $t \in [0, T]$. We have

\[
(v_t \circ \phi_t) \# \nu_0 = v_t \# \sigma_t = (v_t \circ \phi_t) \# \nu_0.
\]

This, together with Theorem 5.17 yields the conclusion. \[\square\]

As a consequence, one can define the functional $\mathcal{F}$ on $AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d))$ as in (1.5). We are now ready to identify the $\Gamma(\tau)$-limit of $\{\mathcal{F}^\epsilon\}$ as $\epsilon \to 0$.

**Theorem 5.19.** Consider $\mathcal{F}$ given by (1.5), where $v$ is the velocity of minimal $L^p(\sigma)$-norm associated with $\sigma$. We have

\[
\mathcal{F} = \Gamma(\tau) \lim_{\epsilon \to 0} \mathcal{F}^\epsilon \quad \text{on } \mathcal{S}_p(\mathbb{R}^d).
\]
**Proof:** Consider an arbitrary sequence of positive numbers \( \{\epsilon_n\} \) that converges to 0. Let \((\sigma, v) \in \mathcal{S}_0(\mathbb{R}^d)\) and take \( \{(\sigma^n, v^n)\} \subset \mathcal{S}_p(\mathbb{R}^d) \) to be any sequence that \( \tau \)-converges to \((\sigma,v)\). Choose \( \{\phi\} \cup \{\phi^n\} \subset \mathcal{AC}^p(0,T;\mathcal{L}^p(X)) \) such that \( \phi \) (resp. \( \phi^n \)) is a flow associated to \((\sigma,v)\) (resp. \((\sigma^n,v^n)\)) such that \( \{\phi^n\} \) \( \tau \)-converges to \( \phi \) in \( \mathcal{AC}^p(0,T;\mathcal{L}^p(X)) \). We have

\[
(5.21) \quad \liminf_{n \to \infty} F^{\epsilon_n}(\sigma^n) = \liminf_{n \to \infty} F^{\epsilon_n}(\phi^n) \geq F(\phi) = \int_0^T \hat{L}(\dot{\phi}_t) dt = \int_0^T \hat{L}(v_t \circ \phi_t) dt = \int_0^T \hat{L}(\sigma_t, v_t) dt,
\]

where we have used Theorem 2.15 and Corollary 5.18.

Next, choose a sequence \( \{\psi^n\} \subset \mathcal{AC}^p(0,T;\mathcal{L}^p(X)) \) that \( \tau \)-converges to \( \phi \) and such that

\[
(5.22) \quad \lim_{n \to \infty} F^{\epsilon_n}(\psi^n) = F(\phi) = \int_0^T \hat{L}(\dot{\phi}_t) dt = \int_0^T \hat{L}(v_t \circ \phi_t) dt.
\]

Since \( \mathcal{W} \) is continuous (cf. Lemma 5.2) and bounded on \( \mathcal{P}_\mu(\mathbb{R}^d) \), thanks to Corollary 2.3 we may assume without loss of generality that \( \phi^n \) is of the form \( \psi^n_t = \sum_{j=1}^{m_n} x_{j,n} \chi_{C_{j,n}} \), where \( \{C_{j,n}\}_{j=1}^{m_n} \) is a partition of \( X \) consisting of same-size squares. Lemma 7.4 provides a family \( \psi^n : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d \) of Borel maps such that \( \psi^n_t = v^n_t \circ \psi^n_t \) for almost every \( t \in (0,T) \) and \( v^n \) is the velocity of minimal norm for \( \sigma^n \in \mathcal{AC}^p(0,T;\mathcal{P}_\mu(\mathbb{R}^d)) \) defined by \( \sigma^n_t = \psi^n_{t#} \nu_0 \). We have that \( \{\sigma^n\} \) \( \tau \)-converges to \( \sigma \) in \( \mathcal{S}_\mu(\mathbb{R}^d) \) and so, combining this fact with (5.22), we obtain

\[
(5.23) \quad \lim_{n \to \infty} F^{\epsilon_n}(\sigma^n) = \lim_{n \to \infty} F^{\epsilon_n}(\psi^n) = \int_0^T \hat{L}(v_t \circ \phi_t) dt = \mathcal{F}(\sigma).
\]

This proves the theorem. \( \square \)

Let \( \sigma \in \mathcal{AC}^p(0,T;\mathcal{P}_\mu(\mathbb{R}^d)) \) and let \( v \) be a velocity for \( \sigma \). Fix \( 0 < s < T/2 \), \( \sigma_0, \sigma_T \in \mathcal{P}_\mu(\mathbb{R}^d) \) and modify \( \sigma \) on \([0,s] \cup [T-s,T]\) to obtain the path \( \bar{\sigma} \in \mathcal{AC}^p(0,T;\mathcal{P}_\mu(\mathbb{R}^d)) \) defined the following way: on \([0,s]\) we set \( \bar{\sigma} \) to be the geodesic of constant speed in \( \mathcal{P}_\mu(\mathbb{R}^d) \) connecting \( \sigma_0 \) to \( \sigma \). Similarly, on \([T-s,T]\) we set \( \bar{\sigma} \) to be the geodesic of constant speed in \( \mathcal{P}_\mu(\mathbb{R}^d) \) connecting \( \sigma_{T-s} \) to \( \bar{\sigma}_T \). We denote by \( \bar{v} \) the geodesic velocity on \([0,s]\) and \([T-s,T]\) and set \( \bar{v} \) to coincide with \( v \) on \([s,T-s]\). We use (5.6) and the fact that \( \mathcal{L} \geq 0 \) to obtain

\[
(5.24) \quad \int_0^T \mathcal{L}(\sigma_t, v_t) dt \geq \int_0^T \mathcal{L}^c(\bar{\sigma}_t, \bar{v}_t) dt - C_s \left( 2 + \frac{W\bar{p}_p(\sigma_0, \sigma_T) + W\bar{p}_p(\sigma_T, \sigma_{T-s})}{s^p} \right).
\]

**Lemma 5.20.** If \( \{\sigma^n_t\} \) converges to \( \sigma_t \) in \( \mathcal{P}_\mu(\mathbb{R}^d) \) for \( t = 0, T \), then

\[
\lim_{n \to \infty} \mathcal{C}^c(\sigma^n_0, \sigma^n_T) \geq \lim_{n \to \infty} \mathcal{C}^c(\sigma_0, \sigma_T).
\]

**Proof:** By Lemma 5.3, there exists \( \{\sigma^n\} \subset \mathcal{AC}^p(0,T;\mathcal{P}_\mu(\mathbb{R}^d)) \) such that \( \mathcal{C}^c(\sigma^n, \sigma^n_T) = F^{\epsilon_n}(\sigma^n) \) and \( E < \infty \) such that \( |(\sigma^n')/(t)| \leq E \) for every \( n \in \mathbb{N} \) and almost every \( t \in (0,T) \). Let \( \sigma^n \) be the sequence obtained by modifying \( \sigma \) as above on \([0,s] \cup [T-s,T] \), so that \( \sigma^n_0 = \sigma_0 \) for \( t = 0, T \). By (5.24),

\[
\liminf_{n \to \infty} \mathcal{C}^c(\sigma^n_0, \sigma^n_T) \geq \liminf_{n \to \infty} \mathcal{F}(\sigma^n) - C_s \left( 2 + \frac{W\bar{p}_p(\sigma_0, \sigma_T) + W\bar{p}_p(\sigma_T, \sigma_{T-s})}{s^p} \right) \geq \liminf_{n \to \infty} \mathcal{C}^c(\sigma_0, \sigma_T) - C_s(2 + 2E^p).
\]

We let \( s \) tend to 0 to conclude the proof. \( \square \)

**Remark 5.21.** Let \( \xi \in \mathcal{L}^p(X) \) and define \( \sigma_t = (t\xi)_{t#} \nu_0 \). Then \( \sigma \in \mathcal{AC}^p(0,T;\mathcal{P}_\mu(\mathbb{R}^d)) \) and \( (t,x) \rightarrow v_t(x) = x/t \) is the velocity of minimal norm for \( \sigma \). Set \( \phi(z) = t\xi(z) \) for \( t \in [0,T] \) and \( z \in X \). We have that \( \phi \in \mathcal{AC}^p(0,T;\mathcal{L}^p(X)) \) and \( \phi \) is a flow associated to \((\sigma,v)\).
Suppose \( \text{Theorem 5.22.} \)

for \( \text{Proof:} \) Fix \( 0 \leq s \leq t \leq T. \) We have

\[
W_p^p(\sigma_t, \delta_0) = t^p ||\xi||^p \quad \text{and} \quad W_p^p(\sigma_t, \sigma_t) \leq (t-s)||\xi||^p,
\]

which proves that \( \sigma \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d)). \)

Setting \( \nu(x) = x/t, \) we obtain that \( \nu \) is a Borel map such that \( t \to ||\nu(t)||_\sigma \) belongs to \( L^p(0, T). \) Furthermore, if \( F \in C_c^\infty((0, T) \times \mathbb{R}^d), \) using that \( \sigma_t = (t\xi)_y \) and that \( \nu(t)(\xi(z)) = \xi(z), \) we obtain

\[
\int_0^T dt \int_{\mathbb{R}^d} \left( \partial_t F + \langle \nabla F; \nu_t \rangle \right) d\sigma_t = \int_0^T dt \int_X \left( \partial_t F(t, \xi(z)) + \langle \nabla F(t, \xi(z)); \nu_t(\xi(z)) \rangle \right) dz
\]

\[
= \int_0^T dt \int_X \frac{d}{dt} F(t, \xi(z)) dz = 0,
\]

which proves that \( \nu \) is a velocity for \( \sigma. \) Set \( \varphi(x) = t \frac{1}{|x|} / |\mu| |x|^p \) so that \( \nabla \varphi |\nabla \varphi|^{p-2} = \nu \in L^p(\sigma_t). \) This proves that \( \nu_t \in \mathcal{T}_\omega \mathcal{P}_p(\mathbb{R}^d) \) and so, \( \nu \) is the velocity of minimal norm for \( \sigma. \) One readily checks that \( \nu_t \circ \varphi = \hat{\phi}_t \) to conclude the proof of the remark. \( \square \)

**Theorem 5.22.** Suppose \( \mathcal{F}_C \Gamma(\tau_w) \)-converges to \( \mathcal{F}_\omega \) and \( \Gamma(\tau) \)-converges to \( \mathcal{F}. \) Let \( \xi \in L^p(X) \) and set \( \sigma_t = (t\xi)_y. \) Then \( \mathcal{F}_\omega(\sigma) = \mathcal{F}(\sigma) = T \mathcal{L}^p(\xi). \)

**Proof:** By Remark 5.11, \( \tau_w \) is weaker than \( \tau \) and so, \( \mathcal{F}_\omega(\sigma) \leq \mathcal{F}(\sigma). \) Set \( \phi_t = D^t \circ \xi. \) By Remark 5.21, \( D^{t-1} \) is the velocity of minimal norm for \( \sigma_t \) and, as \( D^t \circ \xi = t\xi \) pushes \( \nu_0 \) forward to \( \sigma_t, \) we use Theorem 5.19 to conclude that \( \mathcal{F}(\sigma) = \int_0^T \bar{L}(D^{t-1} \circ D^t \circ \xi) dt = T \mathcal{L}(\xi). \) It remains to show the reverse inequality. Let \( \{\sigma^n\}_n \subset AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d)) \) be a sequence that \( \tau_w \)-converges to \( \sigma \) and such that \( \mathcal{F}_\omega(\sigma) = \lim_{n \to \infty} \mathcal{F}_C(\tau) \). By Corollary 5.14, we may assume that \( \lim_{n \to \infty} W_p(\sigma^n, \sigma) = 0 \) for all \( t \in [0, T]. \) Fix \( s \in (0, T) \) and let \( \bar{\sigma}^n \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d)) \) be obtained by modifying \( \sigma^n \) the following way: on \( [0, s], \) \( \bar{\sigma}^n \) is the geodesic of constant speed starting at \( \sigma_0 = \delta_0 \) and ending at \( \sigma^s. \) On \( [s, T-s], \) \( \bar{\sigma}^n \) coincides with \( \sigma^n. \) On \( [T-s, T], \) \( \bar{\sigma}^n \) is the geodesic of constant speed starting at \( \sigma^n_{T-s} \) and ending at \( \sigma_T. \) By (5.24)

\[
(5.25)
\mathcal{F}_\omega(\sigma) \geq \limsup_{n \to \infty} \mathcal{F}_C(\bar{\sigma}^n) - C(2 + e) \geq \limsup_{n \to \infty} C_{0,T/n}(\sigma_0, \sigma_T) - C(2 + e) s,
\]

where we have used that \( W_p(\sigma_0, \sigma) = W_p(\sigma_{T-s}, \sigma_T) = se \) where \( e = \int_X |\xi|^p dz. \) As \( s \in (0, T) \) is arbitrary, we conclude that \( \mathcal{F}(\sigma) \geq \limsup_{n \to \infty} C_{0,T/n}(\sigma_0, \sigma_T). \) We now use (5.4) to conclude that

\[
\mathcal{F}_\omega(\sigma) \geq T \liminf_{n \to \infty} \frac{C_{0,T/n}(\delta_0, D^{T/n} \circ \sigma_T)}{T/n} = T \liminf_{n \to \infty} \frac{C_{0,T/n}(\delta_0, D^{T/n} \circ \xi)}{T/n} \geq T \mathcal{L}(\nu_0, \xi).
\]

This, together with Theorem 5.17, yields the proof. \( \square \)

6. A PARTICULAR CASE OF HOMOGENIZATION ON \( \mathcal{P}_p(\mathbb{R}^d) \)

Throughout this section we assume that \( T > 0 \) and \( 1 < p < \infty, \) \( W \in C(\mathbb{T}^d) \) is even and satisfies \( W(x) \leq W(0) = 0 \) for \( x \in \mathbb{R}^d. \) We set \( X = (0, 1)^d \) and define

\[
L(M, N) = \frac{1}{p} ||N||^p - \int_{X \times X} W(Mz - Mw) dwdz, \quad L'(M, N) = \frac{1}{p} ||N||^p - \int_{X \times X} W \left( \frac{Mz - Mw}{\epsilon} \right) dwdz,
\]

for \( M, N \in L^p(X) \) and \( \epsilon > 0. \) Similarly, we define

\[
\mathcal{L}(\mu, \xi) = \frac{1}{p} ||\xi||_\mu^p - \int_{\mathbb{R}^d \times \mathbb{R}^d} W(x - y) \mu(dx) \mu(dy), \quad \mathcal{L}'(\mu, \xi) = \frac{1}{p} ||\xi||_\mu^p - \int_{\mathbb{R}^d \times \mathbb{R}^d} W \left( \frac{x - y}{\epsilon} \right) \mu(dx) \mu(dy),
\]

for \( \mu \in \mathcal{P}_p(\mathbb{R}^d) \) and \( \xi \in L^p(\mu) \) so that \( \mathcal{L}'(\mu, \xi) = \mathcal{L}(D^1/\epsilon, \mu \circ D^1). \)
Remark 6.1. Note that if \( c \in \mathbb{R} \) and \( \phi \in AC^p(0,T;L^p(X)) \) is such that \( \phi_0 = \phi_T \equiv 0 \), then using that \( W \leq 0 \) we have

\[
\int_0^T L(ct + \phi_t, c + \phi_t)dt \geq \int_0^T L(ct, c)dt = \frac{1}{p} T|c|^p.
\]

Thus, \( \bar{L}(c) = |c|^p / p \). Let \( c^* = c|c|^{p-2} \) and let \( \bar{H} \) be the Legendre transform on \( \bar{L} \). Then

\[
\bar{H}(c^*) \geq -\bar{L}(c) + c^* \cdot c = \frac{1}{p} |c^*|^{p'},
\]

which proves that \( \bar{H}(c^*) = |c^*|^{p'}/p' \).

6.1. Actions restricted to paths consisting of Dirac masses. If \( m \) is a positive integer, we denote by \( \mathcal{P}^m(\mathbb{R}^d) \) the set of measures \( \mu \) which are averages of \( m \) Dirac masses: \( 1/m \sum_{i=1}^m \delta_{x_i} \), where \( \{x_i\}_{i=1}^m \subset \mathbb{R}^d \). For \( \mu_a, \mu_b \in \mathcal{P}^m(\mathbb{R}^d) \) we set

\[
\mathcal{C}_{a,b,m}(\mu_a, \mu_b) = \inf_{\sigma} \left\{ \int_a^b \mathcal{L}(\sigma_t, \nu_t)dt : \sigma \in AC^p(a,b;\mathcal{P}^m(\mathbb{R}^d)), \sigma_a = \mu_a, \sigma_b = \mu_b \right\}.
\]

Following verbatim the proof of Lemma 3.4 [16], one obtains that on \( \mathcal{P}_p^m(\mathbb{R}^d) \times \mathcal{P}_p^m(\mathbb{R}^d) \),

(6.1) \[
\mathcal{C}_{a,b,m} = \mathcal{C}_{a,b,km} \text{ for any integer } k \geq 1.
\]

Lemma 6.2. If \( \mu_a, \mu_b \in \mathcal{P}_p^m(\mathbb{R}^d) \), then \( \mathcal{C}_{a,b,m}(\mu_a, \mu_b) = \mathcal{C}_{a,b}(\mu_a, \mu_b) \).

Proof: We may assume without loss of generality that \( \epsilon = 1 = a = 0 \) and \( b = T \). Let \( \sigma \) be the minimizer of \( \mathcal{F}(\cdot, (0,T)) \) over the set of paths in \( AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d)) \) which have \( \sigma_0 \) and \( \sigma_T \) as endpoints, as provided by Lemma 5.3. It suffices to show that for every \( \delta > 0 \),

\[
\int_0^T \mathcal{L}(\sigma_t, \nu_t)dt \geq \mathcal{C}_{0,T,m}(\mu_0, \mu_T) - \delta.
\]

Lemma 5.3 ensures existence of a constant \( C_\sigma > 1 \) such that \( \|\nu_t\|_{\sigma_t} \leq C_\sigma - 1 \) for almost every \( t \in (0,T) \). Here, \( \nu \) is a velocity for \( \sigma \). We use Remark 5.16 to find a sequence \( \{\sigma^k\}_k \subset AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d)) \) and \( \nu^k \) a velocity for \( \sigma^k \) such that \( \sigma^k \in C^\infty([0,T] \times \mathbb{R}^d), \nu^k \in C^\infty([0,T] \times \mathbb{R}^d;\mathbb{R}^d), \sigma^k_t \to \sigma_t \) in \( \mathcal{P}_p(\mathbb{R}^d) \) for each \( t \in [0,T] \) and

\[
\int_0^T \mathcal{L}(\sigma^k_t, \nu^k_t)dt \leq \int_0^T \mathcal{L}(\sigma_t, \nu_t)dt + \frac{\delta}{4} \text{ for } k \geq k_0.
\]

The following property is also ensured: \( \|\nu^k_t\|_{\sigma^k_t} \leq \|\nu_t\|_{\sigma_t} \leq C_\sigma - 1 \).

Let \( S^k \) and \( M^k \) be as in the proof of Theorem 5.17, so that

\[
M^k \in AC^p(0,T;L^p(X)), \quad \int_0^T \mathcal{L}(\sigma^k_t, \nu^k_t)dt = \int_0^T L(M^k_t, M^k_t)dt, \quad M^k_{t\#} \nu_0 = \sigma^k_t
\]

for all \( t \in [0,T] \) and \( \|M^k\| = \|\nu^k_t\|_{\sigma^k_t} \leq \|\nu_t\|_{\sigma_t} \) for a.e. \( t \in (0,T) \). We use Corollary 2.3 and the fact that \( \mathcal{W} \) is continuous and bounded to obtain a sequence \( \{\tilde{\sigma}^k\}_k \subset AC^p(0,T;L^p(X)) \) such that for each \( k \) fixed there is a partition of \( X \) into finitely many parallel subcubes of same size such that \( \tilde{\sigma}^k_t \) is constant on each subcube and

\[
\int_0^T L(\phi^k_t, \phi^k_t)dt \leq \int_0^T L(M^k_t, M^k_t)dt + \frac{\delta}{4}, \quad \|M^k_t - \phi^k_t\| \leq \frac{1}{k} \text{ for all } t \in [0,T].
\]

For \( k \) fixed, we may assume that the total number of subcubes is of the form \( (l_k m)^d \), where \( l_k \) is a natural number. Set \( \tilde{\sigma}^k_t = \tilde{\sigma}^k_t \circ \phi^k_t \) so that \( \tilde{\sigma}^k \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d)) \). By Lemma 7.4, there exists \( \tilde{\nu}^k \) velocity for \( \tilde{\sigma}^k \) such that \( \tilde{\phi}^k_t = \tilde{\nu}^k_t \circ \phi^k_t \) almost every \( t \in (0,T) \). Fix \( s \in (0,T) \) and let \( \tilde{\sigma}^k \) be defined the following way: for
Let \( t \in [0, s] \) it is the geodesic of constant speed connecting \( \mu_0 \) to \( \bar{\sigma}_k^t \). For \( t \in [s, T - s] \), \( \bar{\sigma}_k^t \) coincides with \( \bar{\sigma}_k^T \). For \( t \in [T - s, T] \), \( \bar{\sigma}_k^t \) is the geodesic of constant speed connecting \( \bar{\sigma}_k^{T-s} \) to \( \mu_T \). By (5.24)

\[
\int_0^T \mathcal{L}(\bar{\sigma}_k^t, \bar{\nu}_k^t) dt \geq \int_0^T \mathcal{L}(\bar{\sigma}_k^t, \bar{\nu}_k^t) dt - C_S \left( 2 + \frac{W_p^p(\mu_0, \bar{\sigma}_k^t) + W_p^p(\mu_T, \bar{\sigma}_k^{T-s})}{s^p} \right)
\]

We use the triangle inequality and the fact that \( \| \bar{\phi}_k^t \| \leq C_\sigma \) to obtain

\[
W_p^p(\mu_0, \bar{\sigma}_k^t) \leq W_p^p(\mu_0, \sigma_0^t) + W_p^p(\sigma_0^t, \bar{\sigma}_k^t) + W_p^p(\bar{\sigma}_k^t, \bar{\sigma}_k^{T-s}) \leq W_p^p(\mu_0, \sigma_0^t) + \| M_k^t - \phi_k^t \| + sC_\sigma.
\]

and so, for \( k \) large enough \( W_p^p(\mu_0, \sigma_k^t) \leq 2sC_\sigma \). Similarly, for \( k \) large enough \( W_p^p(\mu_T, \bar{\sigma}_k^{T-s}) \leq 2sC_\sigma \). We use (6.2) to obtain for \( s \) small enough

\[
\int_0^T \mathcal{L}(\bar{\sigma}_k^t, \bar{\nu}_k^t) dt \geq \int_0^T \mathcal{L}(\bar{\sigma}_k^t, \bar{\nu}_k^t) dt - \frac{\delta}{4} \geq C_\sigma^\epsilon_{0, T, \epsilon}(\mu_0, \mu_T) - \frac{\delta}{4} = C_\sigma^\epsilon_{0, T, \epsilon}(\mu_0, \mu_T) - \frac{\delta}{4},
\]

where we have used (6.1). This finishes the proof.

\[ \square \]

6.2. Lower bound of the \( \Gamma \)-limit on \( P_\rho(\mathbb{R}^d) \). Let \( l, m > 1 \) be integers, let \( \{ x_i^j \} \subset \mathbb{Q}^d \) where \( i = 0, \cdots, 2^l \) and \( j = 1, \cdots, m^d \). Let \( \sigma \in AC_p(0, T; P_\rho(\mathbb{R}^d)) \) be such that \( \sigma_t = \frac{1}{m^d} \sum_{j=1}^{m^d} \delta_{x_i^j}, \) where

\[
x_i^j := x_i^j + \left( \frac{2^l t}{T} - i \right) (x_{i+1}^j - x_i^j)
\]

whenever \( t \in [s_i, s_{i+1}] \).

Here, \( s_i := Ti/2^l \) for \( i = 0, \cdots, 2^l - 1 \).

**Theorem 6.3.** Let \( \nu \) be the unique velocity for \( \sigma \) and let \( \bar{\phi} \) be a flow associated to \( (\sigma, \nu) \). Then

\[ F_\nu(\sigma) = \int_0^T \bar{L}(\nu_t \circ \bar{\phi}_t) dt. \]

**Proof:** Note that \( \sigma \in S_\rho(\mathbb{R}^d) \). By Remark 5.11, \( \tau_\nu \) is weaker than \( \tau \) and so, \( F_\nu(\sigma) \leq F(\sigma) \). It remains to show the reverse inequality.

By Corollary 5.14, there exists a sequence \( \{ \sigma^n \}_n \) \( \tau_\nu \)-converging to \( \sigma \) such that \( \{ \sigma^n \}_n \) converges to \( \sigma_t \) in \( P_\rho(\mathbb{R}^d) \) for every \( t \in [0, T] \) and \( F_\nu(\sigma) = \lim_{n \to \infty} F_\nu(\sigma^n) \). We exploit subsection 7.2 and use the fact that \( W \) is continuous and bounded (cf. Lemma 5.2) to assume without loss of generality that for each \( n \) there exists an integer \( m_n \) such that \( \sigma^n_t \) is of the form \( 1/m_n \sum_{i=1}^{m_n} \delta_{x_i^j} \). By subsection 7.4, we may assume that \( t \to x_{i}^{m_n} \) belongs to \( W^{1,p}(0, T; \mathbb{R}^d) \). By subsection 7.3, there exist a Borel velocity \( \nu^n \) for \( \sigma^n \) and a flow \( \phi^n \in AC^p(0, T, L^p(X)) \) associated to \( (\sigma^n, \nu^n) \). Fix a positive integer \( k > l \). For \( i = 0, \cdots, 2^k \) we

\[
t_i = ih \quad i = 0, \cdots, 2^k, \quad h = \frac{T}{2^k}. \]

Note \( \nu^n \) is the velocity of minimal norm for \( \sigma^n \). We have

\[
F(\sigma) = \lim_{n \to \infty} F_\nu^n(\sigma^n) \leq \lim_{n \to \infty} F_\nu^n(\phi^n) = \lim_{n \to \infty} \sum_{i=0}^{2^k-1} \int_{t_i}^{t_{i+1}} \mathcal{L}^{n}(\sigma^n_t, \nu^n_t) dt
\]

\[
\geq \sum_{i=0}^{2^k-1} \lim_{n \to \infty} C_{i, i+1}^{\epsilon_n}(\sigma^n_t, \sigma^n_{t_{i+1}}) \geq \sum_{i=0}^{2^k-1} \lim_{n \to \infty} C_{i, i+1}^{\epsilon_n}(\sigma_t, \sigma_{t_{i+1}})
\]

\[
= \sum_{i=0}^{2^k-1} \lim_{n \to \infty} \epsilon_n C_{i, i+1}^{\epsilon_n} \left( D^{1/\epsilon_n} \sigma_t, D^{1/\epsilon_n} \sigma_{t_{i+1}} \right).
\]
We have used Lemma 5.20 and the fact that \( \{\sigma^n_i\} \) converges to \( \sigma_t \) for \( t = t_i, t_{i+1} \) to obtain the last expression in (6.4). To obtain (6.5) we have used (2.9). We exploit Lemma 5.3 and then Lemma 6.2 to find a minimizer \( \bar{\sigma}^n \in \mathcal{A}C^p(0,T;\mathcal{P}_p(\mathbb{R}^d)) \) such that for \( t \in [0,T], \) \( \bar{\sigma}^n_t \) is of the form

\[
\bar{\sigma}^n_t = \frac{1}{m^n} \sum_{j=1}^{m^n} \delta_{x_{i,j}^n} \quad \text{and} \quad \bar{\sigma}_t = \sigma_t
\]

for \( i = 0, 1, ..., 2^k - 1 \) and

\[
e_n C_0 \frac{n}{\epsilon_n} \left( D_{\#}^{1/\epsilon_n} \bar{\sigma}_t, D_{\#}^{1/\epsilon_n} \bar{\sigma}_{t+1} \right) = \int_{t_i}^{t_{i+1}} \mathcal{L}(D_{\#}^{1/\epsilon_n} \bar{\sigma}, \psi^n) \, dt,
\]

where \( \psi^n \) is the unique velocity for \( \bar{\sigma}^n \). Thanks to subsection 7.4, we may assume that \( t \to \bar{x}_{t,i}^n \) is of class \( W^{1,p}(0,T;\mathbb{R}^d) \). By Lemma 7.2, there exists a flow \( \bar{\phi}^n \) associated to \( \bar{\sigma}^n \). Furthermore, there are \( m \) parallel subcubes of \( X \) of same size such that for each subcube \( \bar{\phi}^n_i \) is the point \( \bar{x}_{t,i}^n \) for some \( i \). We have

\[
e_n C_0 \frac{n}{\epsilon_n} \left( D_{\#}^{1/\epsilon_n} \bar{\sigma}_t, D_{\#}^{1/\epsilon_n} \bar{\sigma}_{t+1} \right) = \int_{t_i}^{t_{i+1}} \mathcal{L}(\bar{\phi}^n_t, \bar{\phi}^n_t) \, dt = e_n C_0 \frac{n}{\epsilon_n} \left( \frac{\bar{\phi}^n_t}{\epsilon_n}, \bar{\phi}^n_t + 1 \right).
\]

By (2.6), for any \( P > 0 \) we have

\[
C_0, P_{\#}^{\epsilon_n} - C_0, P_{\#}^{\epsilon_n+1} + 1 = C_0, P_{\#}^{\epsilon_n+1} = C_0, P_{\#}^{\epsilon_n+2} \left[ \frac{1}{P_{\#}^{\epsilon_n}} \right] \phi_{t+1}^{n} + 1 \right] \left[ \frac{1}{P_{\#}^{\epsilon_n}} \right] \phi_{t+1}^{n}
\]

and

\[
\geq -6C + C_0, P_{\#}^{\epsilon_n+2} \left( \frac{1}{P_{\#}^{\epsilon_n}} \right) \phi_{t+1}^{n} + 1 \right] \left[ \frac{1}{P_{\#}^{\epsilon_n}} \right] \phi_{t+1}^{n}
\]

We have used (2.1) to obtain (6.7). By Remark 2.1,

\[
C_0, P_{\#}^{\epsilon_n+2} \left[ \frac{1}{P_{\#}^{\epsilon_n}} \right] \phi_{t+1}^{n} + 1 \right] \left[ \frac{1}{P_{\#}^{\epsilon_n}} \right] \phi_{t+1}^{n} \geq \left( 2 + \frac{h}{\epsilon_n} \right) \bar{L}(q_n), \text{ where } q_n = \frac{\left( \frac{1}{P_{\#}^{\epsilon_n}} \right) \phi_{t+1}^{n} - \phi_{t}^{n}}{h + \frac{h}{\epsilon_n}}.
\]

Although \( \phi_{t}^{n} \) depends on \( n \), note that, if \( t_i \in [s_i, s_{i+1}] \), the range of \( \phi_{t}^{n} \) is the subset of finite cardinality \( \{x_{i,j}^n \} \) contained in \( Q^d \). The union of these sets depends on \( k \) but is independent of \( n \). Hence, \( \phi_{t}^{n} \) is bounded in \( L^{\infty}(0,T;L^p(X)) \) and there exists an integer \( P_k \) independent of \( n \) such that the range of \( P_k \phi_{t}^{n} \) is contained in \( Z^d \) for \( i = 0, 1, ..., 2^k - 1 \). We also have

\[
q_n = \frac{P_k \left( \frac{1}{P_{\#}^{\epsilon_n}} - \frac{1}{P_{\#}^{\epsilon_n+1}} \right) \phi_{t+1}^{n} - \phi_{t}^{n}}{h + 2\epsilon_n}.
\]

Hence,

\[
\left\| q_n - \phi_{t+1}^{n} + \phi_{t}^{n} \frac{h + 2\epsilon_n}{h + 2\epsilon_n} \right\| \leq e_n h \frac{\phi_{t+1}^{n} - \phi_{t}^{n}}{h + 2\epsilon_n}.
\]

which implies that \( \{q^n\}_n \) is bounded in \( L^p(X) \). As \( \bar{L} \) is convex and assumes only finite values, it is locally Lipschitz. Up to a subsequence which we do not relabel, \( \{\bar{\phi}^n_t\}_n \) converges in \( L^p(X) \) to some \( \phi^k_t \) which depends on \( k \) and that pushes \( v_0 \) forward to \( \sigma_t \). Furthermore, \( \phi^k_t \) is a constant on each subcube. Observe that \( \lim_{k \to \infty} W_p(\phi^k_{t,i} \# \nu_0, \sigma_t) = 0 \). We combine (6.5)-(6.8) and (6.10) to conclude that

\[
\mathcal{F}(\sigma) \geq \sum_{i=0}^{2^k-1} h \bar{L} \left( \frac{\phi^k_{r,t+i} - \phi^k_{r,t+i+1}}{h} \right) = \int_0^T \bar{L}(\phi^k_t) \, dt.
\]
For $t \in [t_i, t_{i+1}]$ we have set
\[ \phi^k_t = \left( 1 - \frac{t - t_i}{h} \right) \phi^k_{t_i} + \frac{t - t_i}{h} \phi^k_{t_{i+1}}. \]

When $t_i, t_{i+1} \in [s_i, s_{i+1}]$ (and for each $i \in \{0, 1, \ldots, 2^k - 1\}$ there exists such an $i^* \in \{0, 1, \ldots, 2^k - 1\}$), we have
\[ |\phi^k_{t_{i+1}} - \phi^k_{t_i}| \leq 2^k h |x^i_{i+1} - x^i_i|. \]

Hence, $\{\phi^k_t\}_k$ is bounded in $W^{1,\infty}(0,T;L^p(X))$. Let $E > 0$ be a constant such that $|\phi^k'(t)|(t) \leq E$ for almost every $t \in (0,T)$. The sequence $\{\phi^k_t\}_k$ converges strongly in $L^\infty(0,T;L^p(X))$ to some $\phi$ such that $\phi_t$ is constant on each subcube and $|\phi'(t)|(t) \leq E$ for almost every $t \in (0,T)$. Let $\nu_t = \phi_t - \nu_0$ and observe that $\nu_t = \sigma_t$. Let $t \in [0,T)$. For $k$ fixed, we have $t \in [t_i, t_{i+1}]$ for some $i \in \{0, 1, \ldots, 2^k - 1\}$ and
\[ W_p(\sigma_t, \nu_t) \leq W_p(\sigma_t, \sigma_{t_i}) + W_p(\nu_t, \nu_{t_i}) \leq 2E(t - t_i) \leq 2^{1-k} E. \]

This proves that $\nu = \sigma$. Let $v$ be the unique velocity for $\sigma$ so that $\phi$ is a flow associated to $(\sigma, v)$. By (6.11)
\[ F(\sigma) \geq \int_0^T \dot{L}(\phi_t)dt = \int_0^T \dot{L}(v_t \circ \phi_t)dt = \int_0^T L(v_t \circ \phi_t)dt. \]

To obtain the last inequality we have used that $(v_t \circ \phi_t)\#\mu_0 = v_t \#\sigma_t = (v_t \circ \phi_t)\#\nu_0$, that $\dot{L}$ is continuous on $L^p(X)$, and then applied Remark 5.6. $\square$

7. Appendix

7.1. Review of $p$-absolutely continuous curves in $P_p(\mathbb{R}^d)$ (cf. e.g. [2]). For $\mu \in P_p(\mathbb{R}^d)$ the $L^p(\mu)$-closure of $\{\nabla \varphi : |\nabla \varphi|^{q-2} \varphi \in C_0^\infty(\mathbb{R}^d)\}$ is denoted by $T_{\mu}P_p(\mathbb{R}^d)$ where $q = p/(p-1)$. Equivalently, $T_{\mu}P_p(\mathbb{R}^d)$ is the set of $\nu$ such that $|\nabla \varphi|^{p-2} \varphi$ belongs to the $L^p(\mu)$-closure of $\{\nabla \varphi : \varphi \in C_0^\infty(\mathbb{R}^d)\}$.

Let $\sigma \in AC^p(a,b;P_p(\mathbb{R}^d))$. According to [2] section 8.3 there exists a Borel vector field $\nu : (a,b) \times \mathbb{R}^d \to \mathbb{R}^d$ such that
\begin{align*}
(7.1) & \quad \partial_t \sigma + \nabla \cdot (\sigma \nu) = 0 \text{ in } D((a,b) \times \mathbb{R}^d) \\
(7.2) & \quad \nu_t \in L^p(\sigma_t) \text{ for a.e. } t \in (a,b) \\
(7.3) & \quad \int_a^b \|\nu_t\|^p_{L^p(\sigma_t)} dt < \infty.
\end{align*}

We say that $\nu$ is a velocity for $\sigma$. We can choose $\nu$ such that $\nu_t \in T_{\sigma_t}P_p(\mathbb{R}^d)$ for almost every $t \in (0,T)$. In the latter case, $\nu$ is uniquely determined and is called the velocity of minimal norm for $\sigma$, i.e. if $\nu$ is another velocity for $\sigma$, then $\|\nu_t\|_{\sigma_t} \leq \|\nu_t\|_{\sigma_t}$ for almost every $t \in (0,T)$. For almost every $t$, the metric derivative
\[ |\sigma'(t)|(t) : \lim_{h \to 0} \frac{W_p(\sigma_{t+h}, \sigma_t)}{|h|} \]
exists and is equal to the $\| \cdot \|_{\sigma_t}$-norm of the velocity of minimal norm at $t$. Here, $W_p$ stands for the $p$-Wasserstein metric. For $0 \leq s < t \leq T$ and $C_{s,t} := \int_s^T |\sigma'|^p(l)dl$ we have
\[ W_p(\sigma_s, \sigma_t) \leq \int_s^t |\sigma'|^p(l) dl \leq |t - s|^{\frac{1}{p}} C_{s,t}. \]

Given $\mu, \nu \in P_p(\mathbb{R}^d)$ we denote by $P(\mu, \nu)$ the set of $\sigma \in AC^p(a,b;P_p(\mathbb{R}^d))$ such that $\sigma_a = \mu$ and $\sigma_b = \nu$.

**Remark 7.1.** Let $\{\sigma_n\}_n \subset AC^p(0,T;P_p(\mathbb{R}^d))$ and let $v^\mu$ be its velocity of minimal norm. Suppose there exists $C > 0$ such that $\int_0^T \|v^\mu_t\|^p_{L^p(\sigma_n)} dt \leq C p$ for all $n \in \mathbb{N}$. Then
Let $\sigma^p_{n,t}$ be for $s < t \leq T$ we have $W_p(\sigma^p_{n,t}, \sigma^p_{n,s}) \leq C|t - s|^{1/p}$ for all $n \in \mathbb{N}$. The following four assertions are equivalent: (1) $\sup_n W_p(\sigma^p_{n,t}, \delta_0) < \infty$, (2) $\sup_n W_p(\sigma^p_{0,t}, \delta_0) < \infty$, (3) there exists $\{t_n\}_n \subset [0, T]$ such that $\sup_n W_p(\sigma^p_{n,t_n}, \delta_0) < \infty$ and (4) $\{\sigma^n\}_n$ is bounded in $AC^p(0, T; P_p(\mathbb{R}^d))$.

(ii) If $q \in [1, p)$, then $AC^p(0, T; P_p(\mathbb{R}^d))$ compactly embeds in $C^{0,1/p}([0, T]; \mathbb{R}^d)$. More precisely, if $\{\sigma^n\}_n \subset AC^p(0, T; P_p(\mathbb{R}^d))$ is bounded, then there exists an increasing sequence $\{n_k\}_k$ and a curve $\sigma \in AC^p(0, T; P_p(\mathbb{R}))$ such that for every $t \in [0, T]$, $\{\sigma^n_{n_k}\}_k$ converges to $\sigma_t$ in $P_p(\mathbb{R}^d)$.

Proof: The uniform Hölder continuity is given by (7.4). Clearly (4) implies (3). Suppose next that (3) holds. Then

$$W_p(\sigma^p_{n,t}, \delta_0) \leq W_p(\sigma^p_{n,t}, \sigma^p_{n,s}) + W_p(\sigma^p_{n,s}, \delta_0) \leq T \hat{\sigma} + sup_n W_p(\sigma^p_{n,t}, \delta_0),$$

which proves (4). One concludes easily that the four assertions are equivalent. This concludes the proof of (i).

Consider the topology given by the narrow convergence, which is a Hausdorff topology, compatible with $W_p$ in the sense of [2] section 2.1. We apply Proposition 3.3.1 [2] to obtain an increasing sequence $\{n_k\}_k$ and $\sigma \in AC^p(0, T; P_p(\mathbb{R}^d))$ such that $\{\sigma^n_{n_k}\}_k$ converges narrowly to $\sigma$ for almost every $t \in [0, T]$. As $\sup_n W_p(\sigma^p_{n,t}, \delta_0) < \infty$, $\{\sigma^n_{n_k}\}_k$ converges to $\sigma$ in $P_p(\mathbb{R}^d)$ for almost every $t \in [0, T]$. The uniform Hölder continuity of $\sigma^n$ yields that in fact $\{\sigma^n_{n_k}\}_k$ converges to $\sigma_t$ in $P_p(\mathbb{R}^d)$ for every $t \in [0, T]$. $\Box$

7.2. Time discretization of $p$-absolutely continuous curves. Let $\sigma \in AC^p(0, T; P_p(\mathbb{R}^d))$ and $v$ be a velocity for $\sigma$. Set $C^p_{v,\sigma} := \int_0^T |v|^{p}(s)ds$. Fix an integer $n > 1$ and set $t_j = j/nT = ih$ for $j = 0, \cdots, n$. Choose $\gamma_i \in \Gamma(\sigma_{t_i}, \sigma_{t_{i+1}})$ such that

$$W_p(\gamma_{t_i}, \gamma_{t_{i+1}}) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^{p} \gamma_i(dx, dy).$$

Let $\pi_1, \pi_2 : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the projections defined by $\pi_1(x, y) = x$, $\pi_2(x, y) = y$. For $s \in [t_i, t_{i+1}]$ we set

$$\sigma^n_{i} := \left(1 - \frac{t_i}{h} \right) \pi_1 + \frac{t_i}{h} \pi_2,$$

It is well-known that $\sigma^n \in AC^p(0, T; P_p(\mathbb{R}^d))$ and it is a geodesic of constant speed on $[t_i, t_{i+1}]$. Its velocity of minimal norm is $v^n$ satisfying

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} F\left(1 - \frac{t_i}{h} \right) \pi_1 + \frac{t_i}{h} \pi_2; \frac{y - x}{h}\right) \gamma_i(dx, dy) = \int_{\mathbb{R}^d} \langle F; v^n_s \rangle d\sigma^n_s.$$

We compile all the well-known facts in the following lemma (cf. e.g. [2]):

Lemma 7.2. We have $W_p(\sigma_{t}, \sigma^n_{t}) \leq 2h^{1/p} C_{\sigma}$ for $t \in [0, T]$. Furthermore,

$$\int_0^T \|v^n_t\|_{\sigma^n_t} dt \leq C_{\sigma}^p \text{ and } \|\sigma^n_t\|_{\sigma^n_t} = |(\sigma^n)'(t)| \leq \frac{\int_{t_i}^{t_{i+1}} |(\sigma')'(s)| ds}{h}$$

whenever $t_i \leq s < t \leq t_{i+1}$.

Remark 7.3. Lemma 7.2 asserts that $\{\sigma^n_{t}\}_n$ converges to $\sigma_t$ in $P_p(\mathbb{R}^d)$. Hence, if $\mathcal{W} : P_p(\mathbb{R}^d) \rightarrow \mathbb{R}$ is continuous and bounded, the Lebesgue dominated convergence theorem yields

$$\lim_{n \rightarrow \infty} \int_0^T \mathcal{W}(\sigma^n_t) dt = \int_0^T \mathcal{W}(\sigma_t) dt.$$
Fix an \( m > 1 \) and a real number \( \delta > 0 \). For \( i = 1, \ldots, n \) let \( \{x_i^j\}_{j=1}^m \subset \mathbb{R}^d \) and set

\[
\tilde{\sigma}_i := \frac{1}{m} \sum_{j=1}^m \delta_{x_i^j}, \quad \tilde{\gamma}_i = \frac{1}{m} \sum_{j=1}^m \delta_{(x_i^j, x_i^{j+1})}.
\]

Assume

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \, (dx, dy) = W^p_p(\tilde{\sigma}_i, \tilde{\sigma}_{i+1}), \quad W^p_p(\sigma_i, \tilde{\sigma}_i) \leq \tilde{\delta}^p
\]

where \( \tilde{\sigma} \) is to be chosen soon. As above, we define

\[
\tilde{\sigma}_i^* := \left( (1 - t - t_i) \pi_1 + \frac{t - t_i}{h} \right) \tilde{\gamma}_i.
\]

Without repeating once the above listed facts, we have \( \tilde{\sigma}^n \in AC^p(0, T; P_p(\mathbb{R}^d)) \) and it is a geodesic of constant speed on \([t_i, t_{i+1}]\). Let \( \psi^n \) be the velocity of minimal norm for \( \tilde{\sigma}^n \). Then

\[
\|\psi^n\|_{\sigma^n} = (\|\tilde{\sigma}^n\|/(t) = W^p_p(\tilde{\sigma}_i, \tilde{\sigma}_{i+1})/h
\]

and so,

\[
\int_0^T \|\psi^n\|^p_{\sigma^n} \, dt = h \sum_{i=1}^n \frac{W^p_p(\tilde{\sigma}_i, \tilde{\sigma}_{i+1})}{h^p}.
\]

We choose \( \tilde{\delta} < \min\{\delta, hC_\sigma\} \) small enough and \( m \) large enough, so that

\[
W^p_p(\tilde{\sigma}_i, \tilde{\sigma}_{i+1}) \leq W^p_p(\sigma_i, \sigma_{i+1}) + h^{p-1} \delta/n
\]

to conclude that

\[
(7.6) \quad \int_0^T \|\psi^n\|^p_{\sigma^n} \, dt \leq h \sum_{i=1}^n \frac{W^p_p(\sigma_i, \sigma_{i+1})}{h^p} + \delta = \int_0^T \|\psi^n\|^p_{\sigma^n} \, dt + \delta.
\]

For \( t \in [t_i, t_{i+1}] \) we have

\[
W^p_p(\tilde{\sigma}^n_i, \sigma^n_i) \leq W^p_p(\tilde{\sigma}^n_i, \tilde{\sigma}^n_i) + W^p_p(\tilde{\sigma}^n_i, \sigma^n_i) + W^p_p(\sigma^n_i, \sigma^n_i)
\]

and so,

\[
(7.7) \quad W^p_p(\tilde{\sigma}^n_i, \sigma^n_i) \leq 3\delta + 2W^p_p(\sigma^n_i, \sigma^n_{i+1}) \leq 5hC_\sigma.
\]

### 7.3. Eulerian and Lagrangian coordinates for spatially discrete flows

Suppose \( C^1, \ldots, C^n \) is a partition of \( X \) up to a negligible set and \( x^1, \ldots, x^n \in W^{1, p}(0, T; \mathbb{R}^d) \). Set

\[
\phi_t = \sum_{i=1}^n x_i^j \chi_{C^i}, \quad \sigma_t = \frac{1}{n} \sum_{i=1}^n \delta_{x_i^j}.
\]

Note that \( \sigma_t = \phi_t \circ \nu_0 \).

**Lemma 7.4.** (i) We have \( \phi \in AC^p(0, T; L^p(X)) \) and there exists a Borel map \( \nu : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d \) such that \( \phi_t = \nu_t \circ \phi_t \) for almost every \( t \in (0, T) \).

(ii) We have \( \sigma \in AC^p(0, T; \mathcal{P}_p(\mathbb{R}^d)) \) and \( \nu \) is the unique velocity for \( \sigma \) up to a negligible set. In other words, \( \phi \) is a flow associated to \( (\sigma, \nu) \).

**Proof:** (i) The fact that \( \phi \in AC^p(0, T; L^p(X)) \) is straightforward to establish. The set of \( t \in (0, T) \) such that each one of the \( x^i \) is differentiable at \( t \) and \( x^j_t = x^j_0 \) implies \( \bar{x}^i_t = \bar{x}^i_0 \) for all \( i, j \) is a set of full measure in \( (0, T) \) (cf. e.g. [8]). Thus, there exists a Borel set \( \mathcal{A} := \mathcal{T} \times X \subset (0, T) \times X \) of full \( L^{1+q}(0, T) \times X \) measure such that

\[
\phi_t(x) = \phi_t(y) \quad \text{implies} \quad \hat{\phi}_t(x) = \hat{\phi}_t(y) \quad \text{whenever} \quad (t, x), (t, y) \in \mathcal{A}.
\]

By Corollary 2.3 [20], there exists a Borel map \( \nu : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d \) such that \( \hat{\phi}_t = \nu \circ \phi_t \) for almost every \( t \in (0, T) \).
(ii) As \( \sigma_t = \phi_t \# \nu_0 \), we have for \( s, t \in [0, T] \) that \( W_p(\sigma_t, \sigma_s) \leq \| \phi_t - \phi_s \| \) and so, since \( \phi \in AcP^p(0, T; L^p(X)) \), we conclude that \( \sigma \in AcP^p(0, T; \mathcal{P}_p(\mathbb{R}^d)) \). Furthermore, using that \( \phi_t = \nu_t \circ \phi_t \), we obtain for \( F \in C_c^\infty((0, T] \times \mathbb{R}^d) \)

\[
\int_0^T dt \int_{\mathbb{R}^d} \left( \partial_t F_t + \langle \nabla F_t; \nu_t \rangle \right) d\sigma_t = \int_0^T dt \int_{\mathbb{R}^d} \left( \partial_t F_t \circ \phi_t + \langle \nabla F_t \circ \phi_t; \phi_t \rangle \right) d\nu_0 = \int_{\mathbb{R}^d} \left( F_T \circ \phi_T - F_0 \circ \phi_0 \right) d\nu_0 = 0.
\]

This proves that \( \nu \) is a velocity for \( \sigma \). If \( w \) is another velocity for \( \sigma \) we have \( \nabla \cdot ((\nu_t - w_t)\sigma_t) = 0 \) for for almost every \( t \in (0, T) \). But for each \( t \in [0, T] \), \( \sigma_t \) is supported by finitely many points and so, the closure of \( \{ \nabla \varphi : \varphi \in C_c^\infty(\mathbb{R}^d) \} \) equals \( L^p(\sigma_t) \) (cf., e.g., Example 2.8 [11]). This proves that \( w_t = \nu_t \sigma_t \)-almost everywhere. Hence, \( \nu = w \) up to a negligible set.

\[
\square
\]

7.4. A time regularity result for spatially discrete \( p \)-absolutely continuous curves in \( \mathcal{P}_p(\mathbb{R}^d) \). As in subsection 7.2, let \( \sigma \in AcP^p(0, T; \mathcal{P}_p(\mathbb{R}^d)) \) and let \( \nu \) be a velocity for \( \sigma \). Suppose in addition that there exists \( x^1, \ldots, x^n : [0, T] \to \mathbb{R}^d \) such that \( \sigma_t = 1/n \sum_{j=1}^n \delta_{x_j(t)} \).

Note that, if for each \( t \in (0, T) \) we choose a permutation of \( n \) letters \( \tau(t, \cdot) \), then \( \sigma_t = \sum_{j=1}^n \delta_{x_j(t)} \tau(t, \cdot) \). Unless \( \tau(t, \cdot) \) is chosen appropriately, \( t \to x^{\tau(t, \cdot)}(t) \) may not be in \( W^{1,p}(0, T; \mathbb{R}^d) \). For instance, let \( x_t = t \) for all rational \( t \in [0, T] \) and \( x_t = 1 - t \) for all irrational \( t \in [0, T] \). Similarly, let \( y_t = t \) for all irrational \( t \in [0, T] \) and \( y_t = 1 - t \) for all rational \( t \in [0, T] \). We have \( t \to 1/(\delta_{x_t} + \delta_{y_t}) \) belongs to \( AcP^p(0, T; \mathcal{P}_p(\mathbb{R})) \) whereas neither \( x \) nor \( y \) is continuous. The goal of this section is to show that for an appropriate choice of \( \tau \), \( t \to x^{\tau(t, \cdot)}(t) \) is in \( W^{1,p}(0, T; \mathbb{R}^d) \). Combining this with subsection 7.3 we obtain existence of a flow \( \phi \) associated to \( (\sigma, \nu) \). In other words, \( \phi \in AcP^p(0, T; L^p(X)) \) and \( \phi_t = \nu_t \circ \phi_t \) for almost every \( t \in (0, T) \).

Let \( \sigma^N \) and \( \gamma_i^N \) be as in subsection 7.2. We start ordering the set \( \{x_0\}_j=1 \) anyway we want. The standard mass transportation theory ensures we can choose an ordering of \( \{x_i^j\}_j=1 \), \( i = 0, ..., N \) such that \( \gamma_i^N \) is of the form \( \gamma_i^N = \frac{1}{n} \sum_{j=1}^n \delta_{x_i^j} \). For \( t \in [t_i, t_{i+1}] \) we set

\[
x_t^j = \left( 1 - \frac{t - t_i}{h} \right) x_{t_i} + \frac{t - t_i}{h} x_{t_{i+1}}.
\]

We denote by \( \phi^N(t) \) the vector in \( \mathbb{R}^{nd} \) whose \( j \) component is \( x_t^j \) and by \( \| \cdot \|_{nd} \) the euclidean norm on \( \mathbb{R}^{nd} \). We have \( \phi^N \in W^{1,p}(0, T; \mathbb{R}^{nd}) \) and by (7.5)

\[
\|\phi^N\|_{nd}^p \leq n \int_0^T \|\sigma^N\|^p_\text{pd} \leq \int_0^T |\sigma'|^p |s| ds.
\]

Hence there exists a subsequence \( \{\Phi^N_k\}_k \) that converges weakly to some \( \phi \) in \( W^{1,p}(0, T; \mathbb{R}^{nd}) \), converges strongly in \( L^p(0, T; \mathbb{R}^{nd}) \) and converges pointwise in \( (0, T) \). Thus \( \{\sigma_k^N\}_k \) converges in \( \mathcal{P}_p(\mathbb{R}^d) \) to

\[
\tilde{\sigma}_t := \frac{1}{n} \sum_{j=1}^n \delta_{\phi_t^j}.
\]

Lemma 7.2 gives that \( \sigma_t = \tilde{\sigma}_t \).

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