**Compute derivatives of implicit functions**

**Facts:** An equation $F(x, y) = 0$ involving variables $x$ and $y$ ( may define $y$ as a function $y = y(x)$. To compute $y' = \frac{dy}{dx}$, one can apply the following procedure.

(Step 1) View $y = y(x)$ and differentiate both sides of the equation $F(x, y) = 0$ with respect to $x$. This will yield a new equation involving $x$, $y$ and $y'$.
(Step 2) Solve the resulting equation from (Step 1) for $y'$.

**Example 1** Given $x^4 + x^2y^2 + y^4 = 48$, find $\frac{dy}{dx}$.

**Solution**: View $y = y(x)$ and differentiate both sides of the equation $x^4 + x^2y^2 + y^4 = 48$ to get

$$4x^3 + 2xy^2 + 2x^2yy' + 4y^3y' = 0.$$

To solve this new equation for $y'$, we first combine those terms involving $y'$,

$$(2x^2y + 4y^3)y' = -4x^3 - 2xy^2,$$

and then solve for $y'$:

$$y' = \frac{-4x^3 - 2xy^2}{2x^2y + 4y^3}.$$

**Example 2** Find an equation of line tangent to the curve $xy^2 + x^2y = 2$ at the point $(1, -2)$.

**Solution**: The slope $m$ of this line, is $\frac{dy}{dx}$ at $(1, -2)$, and so we need to find $y'$ first. Apply implicit differentiation. We differentiate both sides of the equation $xy^2 + x^2y = 2$ with respect to $x$ (view $y = y(x)$ in the process) to get

$$y^2 + 2xyy' + 2xy + x^2y' = 0.$$

Then we solve for $y'$. First we have $(2xy + x^2)y' = -y^2 - 2xy$, and then

$$y' = \frac{-y^2 - 2xy}{2xy + x^2}.$$

At $(1, -2)$, we substitute $x = 1$ and $y = -2$ in $y'$ to get the slope $m = \frac{-(-2)^2 - 2(1)(-2)}{2(1)(-2) + 1^2} = 0$, and so the tangent line is $y = -2$.

**Example 3** Find all the points on the graph of $x^2 + y^2 = 4x + 4y$ at which the tangent line is horizontal.

**Solution**: First find $y'$. We differentiate both sides of the equation $x^2 + y^2 = 4x + 4y$ with respect to $x$ (view $y = y(x)$ in the process) to get

$$2x + 2yy' = 4 + 4y'.$$

Then we solve for $y'$. First we have $(2y - 4)y' = 4 - 2x$, and then

$$y' = \frac{2 - x}{y - 2}.$$

Note that when $x = 2$, the equation $x^2 + y^2 = 4x + 4y$ becomes $4 + y^2 = 8 + 4y$, or $y^2 - 4y = 4$. Solve this equation we get $y = 2 + \sqrt{8}$ and $y = 2 - \sqrt{8}$. Therefore, at $(2, 2 - \sqrt{8})$ and $(2, 2 + \sqrt{8})$, the curve has horizontal tangent lines.