Find derivatives by using the chain rule

Chain Rule:

Suppose that function $g(x)$ is differentiable at x and $f(u)$ is differentiable at $u = g(x)$, then the composition function $h(x) = f(g(x))$ is also differentiable at x and

$$h'(x) = f'(g(x))g'(x).$$

Example 1 Find the derivative of $h(x) = (2x^2 - x + 1)^5$.

Solution: View $u = g(x) = 2x^2 - x + 1$ and $f(u) = u^5$. Then $h(x) = f(g(x))$ and so apply the chain rule to get

$$h'(x) = 5u^4(4x - 1) = 5(2x^2 - x + 1)^4(4x - 1).$$

Example 2 Find the derivative of $h(x) = \left(\frac{x+1}{x-1}\right)^7$.

Solution: View $u = g(x) = \frac{x+1}{x-1}$ and $f(u) = u^7$. Then $h(x) = f(g(x))$ and so apply the chain rule to get

$$h'(x) = 7u^6 \frac{1(x-1) - 1(x+1)}{(x-1)^2} = 7\left(\frac{x+1}{x-1}\right)^6 \frac{-2}{(x-1)^2} = \frac{-14(x+1)^6}{(x-1)^8}.$$

Example 3 Find the derivative of $h(x) = \left[x - \left(1 - \frac{1}{x}\right)^{-1}\right]^{-2}$.

Solution: It would be better to simplify the function first to make the computation easier.

$$h(x) = \left[x - \left(\frac{x-1}{x}\right)^{-1}\right]^{-2} = \left[x - \frac{x}{x-1}\right]^{-2} = \left[\frac{x(x-1) - x}{x-1}\right]^{-2} = \left[\frac{x^2 - 2x}{x-1}\right]^{-2} = \left[\frac{x-1}{x^2 - 2x}\right]^{2}.$$

View $u = g(x) = \frac{x-1}{x^2 - 2x}$ and $f(u) = u^2$. Then $h(x) = f(g(x))$ and so apply the chain rule to get

$$h'(x) = 2u\frac{1(x^2 - 2x) - 2(x-1)^2}{(x^2 - 2x)^2} = 2\left(\frac{x-1}{x^2 - 2x}\right)\frac{-x^2 + 2x - 2}{(x^2 - 2x)^2}.$$

Example 4 Given: $G(t) = f(h(t)), h(1) = 4, f'(4) = 3$, and $h'(1) = -6$. Find $G'(1)$.

Solution: Using the chain rule, we get

$$G'(t) = f'(h(t))h'(t).$$

When $t = 1$, we are given $h(1) = 4, f'(4) = 3$, and $h'(1) = -6$. Substituting these given data, we have the answer

$$G'(1) = f'(h(1))h'(1) = f'(4)h'(1) = 3(-6) = -18.$$