Setting up equalities involving geometry

In problems involving geometry, we often need to express one quantity in terms of others. This is quite common in related rate problems and maximum/mionimum problems in calculus. We here will review the skills to tackle these problems.

What should we look for?

Here, we should seek the relations between the related quantities. See examples below.

Example 1 The length of a rectangle is 6 feet longer than the width. If the perimeter of the rectangle is x feet long, express the width in terms of the perimeter x.

Solution: If W and L represent the width and the length of the rectangle, respectively, then the perimeter $x = 2W + 2L$. (This is the relationship we are looking for!)

Once the relationship has been identified, we can start expressing W in terms of x. As L is 6 longer than W, we know that $L = W + 6$. Substitute $L = W + 6$ into the relationship $x = 2W + 2L$ to get $x = 2W + 2(W + 6)$. Simplifying, we have $x = 4W + 12$. Now solve for W to get $W = \frac{x}{4} - 3$.

Exercise 1: An equilateral triangle (a triangle with three equal sides) has a perimeter x feet. Express the length of one side of the triangle in terms of the perimeter. (Relation: $x = 3L$. Answer: $L = x/3$).

Exercise 2: The length of a rectangle is four inches more than twice the width. If the perimeter of the rectangle is x feet long, express the width in terms of the perimeter x. (Relation: $x = 2W + 2(2W + 4) = 6W + 8$. Answer: $W = (x - 8)/6$).

Exercise 3: The length of a rectangle is four inches more than twice the width. If the perimeter of the rectangle is x feet long, express the length in terms of the perimeter x. (If $L = 2W+4$, then $W = (L-4)/2$. Relation: $x = 2L+2W = 2L+2\cdot(L-4)/2 = 3L-4$. Answer: $L = (x+4)/3$).

Example 2 The length of a rectangle is twice the width. If the area of the rectangle is x square feet, express the width in terms of the area x.

Solution: If W and L represent the width and the length of the rectangle, respectively, then the area $x = WL$. (This is the relationship we are looking for!)

Once the relationship has been identified, we can start expressing W in terms of x. As L is twice W, we know that $L = 2W$. Substitute $L = 2W$ into the relationship $x = WL$ to get $x = 2W^2$. As $W \geq 0$, we can solve for W to get $W = \sqrt{x/2}$.

Exercise 4: The length of a rectangle is four times the width. If the area of the rectangle is x square feet, express the width in terms of the area x. (Relation: $x = LW = 4W^2$. Answer: $W = \sqrt{x/4}$.)

Exercise 5: A rectangle has area x square inches. The length L of the rectangle is 4 inches longer than three times the width. Express the width W in terms of area x. (Relation: $x = LW = (3W+4)W$, Answer: $W = \frac{-4+\sqrt{16+12x}}{6}$. Use quadratic formula and the fact that $W \geq 0$. This one is a bit tricky.)

Example 3 The ratio of the length of a rectangle to the width is 5 to 3. If the perimeter of the rectangle is x feet long, express the width in terms of the perimeter x.

Solution: If W and L represent the width and the length of the rectangle, respectively, then the perimeter $x = 2W + 2L$. (This is the relationship we are looking for!)

Once the relationship has been identified, we can start expressing W in terms of x. As the ratio of the length of a rectangle to the width is 5 to 3, we know that $L/W = 5/3$, or $L = 5W/3$. Substitute $L = 5W/3$ into the relationship $x = 2W + 2L$ to get $x = 2W + 2(5W/3)$. Simplifying, we have $x = 16W/3$. Now solve for W to get $W = \frac{3x}{16}$.

Example 4 A manufacturer is to design an open top box having a square base. If the box have surface area 108 square inches, express the height h in terms of the length of one side of the base square.

Solution: Let x denote the length of one side of the square base. (Draw a picture to help you to understand the situation). Then the surface will have 4 rectangles (four sides) and a square (the base), and so $108 = 4xh + x^2$. (This is the relationship we are looking for!) Now solve $108 = 4xh + x^2$ for h to get $h = \frac{108-x^2}{4x}$.

Example 5 (This is related to the minimum-maximum problems in Calculus.) A manufacturer is to design an open top box having a square base and a surface area 108 square inches. Express the volume of this box in terms of the length of one side of the base square.

Solution: Let x denote the length of one side of the square base, and h the height of the box. Then the volume $V =$ (base area) $\cdot h = x^2h$. (This is the relationship we are looking for!)

From this relationship $V = x^2h$, we recognize that we need to express h in terms of x. With the result in Example 4, we know that $h = \frac{108-x^2}{4x}$. Thus $V = x^2 \cdot \frac{108-x^2}{4x} = \frac{x(108-x^2)}{4}$.

Exercise 6: A manufacturer is to design an open top box having a square base and a surface area 432 square inches. Express the volume of this box in terms of x, the length of one side of the base square. (Relations: $V = x^2 h$ and $h = \frac{432 - x^2}{4x}$. Answer: $V = \frac{x(432 - x^2)}{4}$.)

Exercise 7: A manufacturer is to design an open top box having a square base and a volume 1000 cubic inches. Express the surface area of this box in terms of x, the length of one side of the base square. (Relations: $S = x^2 + 4xh$ and, from $V = x^2 h$, $h = \frac{1000}{x^2}$. Answer: $S = x^2 + \frac{4000}{x}$.)