Determine the relationship of a line and a plane

Example(1): Given a line L and a plane $\mathcal{P}$:

$$\begin{aligned} L: &\quad x = 15 + 7t, y = 10 + 12t, z = 5 - 4t \\ \mathcal{P}: &\quad 12x - 5y + 6z = 10. \end{aligned}$$

Determine if L and $\mathcal{P}$ intersect or are parallel.

Solution: The normal vector of $\mathcal{P}$ is $\mathbf{n} = (12, -5, 6)$ and the line L is parallel to $\mathbf{v} = (7, 12, -4)$. As the dot product $\mathbf{n} \cdot \mathbf{v} = (7)(12) + (-5)(12) + 6(-4) = 84 - 60 - 24 = 0$, L and $\mathcal{P}$ are parallel to each other.

We need to see if the two objects intersect. Since L is parallel to $\mathcal{P}$, if L intersects $\mathcal{P}$, then any point of L should be in $\mathcal{P}$. Thus we only have to check if, when $t = 0$, the point $(15, 10, 5)$ is on $\mathcal{P}$. Note that

$$12(15) - 5(10) + 6(6) = 180 - 50 + 36 \neq 10,$$

and so L and $\mathcal{P}$ do not intersect.

Example(2): Given a line L and a plane $\mathcal{P}$:

$$\begin{aligned} L: &\quad x = 15 - 3t, y = 6 - 5t, z = 2 + 3t \\ \mathcal{P}: &\quad 2x + 3y + 4z = 20. \end{aligned}$$

Determine if L and $\mathcal{P}$ intersect of are parallel.

Solution: The normal vector of $\mathcal{P}$ is $\mathbf{n} = (2, 3, 4)$ and the line L is parallel to $\mathbf{v} = (-3, -5, 3)$. As the dot product $\mathbf{n} \cdot \mathbf{v} = (2)(-3) + (3)(-5) + 4(3) = -6 - 15 + 12 \neq 0$, L and $\mathcal{P}$ are not parallel to each other.

We need to see where the two objects intersect. Substitute the parametric equations of L into the equation of $\mathcal{P}$ and solve the resulting equation for t:

$$2(15 - 3t) + 3(6 - 5t) + 4(2 + 3t) = 20 \Longrightarrow -9t = -36 \Longrightarrow t = 4.$$

Substitute $t = 4$ in the parametric equations of L to get the point of intersection at $(3, -14, 14)$.

Find the angle between two planes

Example: Find the angle between the two planes with equations $2x - y + z = 5$ and $x + y - z = 1$, respectively.

Solution: The planes have normal vectors $\mathbf{a} = (2, -1, 1)$ and $\mathbf{b} = (1, 1, -1)$, respectively. Thus

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = 0, \text{ and so } \theta = \frac{\pi}{2}.$$