Write the equation of a plane

Example(1): Write an equation of a plane through $P(1,0,-1)$ with normal vector $\mathbf{n} = (2,2,-1)$.

Solution: The equation is

$$2(x-1)+2(y-0)+(-1)(z+1)=0.$$

Example(2): Write an equation of a plane through $P(5,1,4)$ and parallel to the plane with equation $x+y-2z=0$.

Solution: Thus the two planes has parallel normal vectors, and so we can use $\mathbf{n} = (1,1,-2)$ as a normal vector of both planes. The answer is

$$(x-5)+(y-1)+(-2)(z-4)=0.$$

Example(3): Write an equation of a plane through $A(1,0,-1)$ $B(3,3,2)$ and $C(4,5,-1)$.

Solution: Let $\mathbf{a} = \overline{AB} = (2,3,3)$ and $\mathbf{b} = \overline{AC} = (3,5,0)$. Then as $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, it can serve as a normal vector of the plane. Note that

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 3 \\ 3 & 5 & 0 \end{vmatrix} = (3-0, -(0-9), 10-9) = (3,9,1).$$

Thus the equation of the plane is

$$3(x-1)+9(y-0)+(z+1)=0.$$

Example(4): Write an equation of a plane through $A(1,-2,3)$ and containing the line $L : x = 3-2t, y = 2+4t, z = 5-4t$.

Solution: Set $t=0$ and $t=1$, we obtained two more points $B(3,2,5)$ and $C(1,6,1)$ on the plane and so this problem becomes the same type of problem as Example(3) above. Let $\mathbf{a} = \overline{AB} = (2,4,2) = 2(1,2,1)$ and $\mathbf{b} = \overline{AC} = (0,8,-2) = 2(0,4,-1)$. Then as $\mathbf{n} = \frac{1}{2}\mathbf{a} \times \frac{1}{2}\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, it can serve as a normal vector of the plane. Note that

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & 4 & -1 \end{vmatrix} = (-2-4, -(-1-0), 4-0) = (-6,1,4).$$

Thus the equation of the plane is

$$-6(x-1)+(y+2)+4(z-3)=0.$$