Compute the cross (vector) product of two vectors

Example: Given $\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, compute $\mathbf{a} \times \mathbf{b}$.

Solution: $\mathbf{a} \times \mathbf{b}$ is equal to

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -2 & 3 & 1 \end{vmatrix} = (-1-9, -(1+6), 3-2) = (-10, -7, 1).$$

Find unit vectors perpendicular to two given vectors

Example: Given $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, find two unit vectors perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Solution: Note that $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. First compute

$$\mathbf{a} \times \mathbf{b} = (10-9, -(5-6), 3-4) = (1, 1, -1).$$

Thus the two desired unit vectors are

$$\frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) \text{ and } \left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

Compute areas

Example: Find the area of the triangle with vertices $P(1,1,0)$, $Q(1,0,1)$ and $R(0,1,1)$.

Solution: Set $\mathbf{a} = \overline{PQ} = (0,-1,1)$ and $\mathbf{b} = \overline{PR} = (-1,0,1)$. Thus the area is

$$\frac{|\mathbf{a} \times \mathbf{b}|}{2} = \frac{|(-1,-1,-1)}{|}2 = \frac{\sqrt{3}}{2}.$$

Compute volumes

Example: Find the volume of the parallelepiped with adjacent edges $\overline{OP}, \overline{OQ}$ and $\overline{OR}$, where $P(1,1,0)$, $Q(1,0,1)$ and $R(0,1,1)$ are three points. Also find the volume of the pyramid with vertices O, P, Q and R.

Solution: Set $\mathbf{a} = \overline{OP} = (1,1,0)$, $\mathbf{b} = \overline{OQ} = (1,0,1)$ and $\mathbf{c} = \overline{OR} = (0,1,1)$. Then the volume of the parallelepiped is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$. First compute

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0 - (0+1+1) = -2.$$

Thus the volume of the parallelepiped is 2.

The volume of the pyramid is one sixth of the volume of the parallelepiped, and so it $\frac{1}{3}$.