Compute integrals using Green's Theorem

Useful facts: For a region R on the plane enclosed by a piecewise smooth curve C, where the positive direction of C is the counterclockwise direction.
(i) Green's Theorem states that if $P(x,y)$ and $Q(x,y)$ have continuous first order of derivatives, then

$$\oint_C Pdx + Qdy = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA,$$

where the integral is taken along the positive direction of C.
(ii) The area of the region R is

$$A = \frac{1}{2}\oint_C (-y)dx + xdy = -\oint ydx = \oint xdy.$$

(iii) If $\mathbf{n}$ denotes the outer unit normal vector of the closed curve C, and $\mathbf{F}$ is a vector field, then the **flux of the vector field F across the curve** C is

$$\oint_C \mathbf{F}\cdot\mathbf{n}ds.$$

(iv) With Green's Theorem,

$$\oint_C \mathbf{F}\cdot\mathbf{n}ds = \int\int_R \nabla\cdot\mathbf{F}dA.$$

Example (1) Given $P = x^2 + y^2, Q = -2xy$, C is the boundary of the triangle bounded by $x = 0$, $y = 0$ and $x + y = 1$, Compute $\oint_C Pdx + Qdy$.

Solution: Let R denote the region bounded by C. Apply Green's Theorem to get

$$\oint_C Pdx + Qdy = \int\int_R (-2y - 2y)dA = -4\int_0^1\int_0^{1-y} ydxdy = \frac{-2}{3}.$$

Example (2) Given $P = y^2, Q = 2x - 3y$, C is the $x^2 + y^2 = 9$, Compute $\oint_C Pdx + Qdy$.

Solution: Let R denote the region bounded by C. Apply Green's Theorem to get

$$\oint_C Pdx + Qdy = \int\int_R (2 - 2y)dA = \int_0^{2\pi}\int_0^3 (2 - 2r\sin\theta)rdrd\theta = 18\pi.$$

Example (3) Given $P = y/(1 + x^2), Q = \tan^{-1}x$, C is the $x^4 + y^4 = 1$, Compute $\oint_C Pdx + Qdy$.

Solution: Let R denote the region bounded by C. Apply Green's Theorem to get

$$\oint_C Pdx + Qdy = \int\int_R 0dA = 0.$$

Example (4) Find the area of the region between the graphs $y = x^2$ and $y = x^3$.

Solution: We write this curve C as the union of the reverse curve of C_1 and the positive direction of C_2, where $C_1 : x = t, y = t^2$ with $0 \le t \le 1$ and $C_2 : x = t, y = t^3$ with $0 \le t \le 1$. Therefore,

$$A = \oint_C x dy = -\int_0^1 2t^2 dt + \int_0^1 3t^3 dt = -\frac{2}{3} + \frac{3}{4} = \frac{5}{12}.$$

Example (5) Find the area of the region R which is between the x-axis and one arch of the cycloid with parametric equations $x = a(t - \sin t)$ and $y = a(1 - \cos t)$.

Solution: We write this curve C as the union of the reverse curve of C_1 and the positive direction of C_2, where $C_1 : x = a(t - \sin t), y = a(1 - \cos t)$ with $0 \le t \le 2\pi$ and $C_2 : x = t, y = 0$ with $0 \le t \le 2\pi a$. Therefore,

$$A = \oint_C x dy = -\int_0^{2\pi} (a^2(t - \sin t)(\sin t) dt + \int_0^{2\pi a} 0 dt = 3\pi a^2.$$

Example (6) Use Green's Theorem to compute the work $W = \oint_C \mathbf{F} \cdot \mathbf{T} ds$, where $\mathbf{F} = (-2y, 3x)$ and C has equation $x^2/9 + y^2/4 = 1$.

Solution: Let R denote the region enclosed by C. Note that $Q_x - P_y = 3 + 2 = 5$, and the area of the ellipse with equation $x^2/a^2 + y^2/b^2 = 1$ is πab. Therefore, by Green's Theorem,

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds = \int\int_R 5 dA = 5\pi(3)(2) = 30\pi.$$

Example (7) Use Green's Theorem to compute the out flux $\phi = \oint_C \mathbf{F} \cdot \mathbf{n} ds$, where $\mathbf{F} = (2x, 3y)$ and C has equation $x^2/9 + y^2/4 = 1$.

Solution: Compute to get $\nabla \cdot \mathbf{F} = 2 + 3 = 5$. With the same integration idea in Example (6), we have

$$\oint_C \mathbf{F} \cdot \mathbf{n} ds = \int\int_R 5 dA = 5\pi(3)(2) = 30\pi.$$