Compute the line integrals in conservative fields and search for potential functions

Useful facts: Let $\mathbf{F} = (P, Q, R)$ denote a vector field. $\mathbf{F}$ is **conservative** if there exists a function f such that $\nabla f = \mathbf{F}$. In this case f is a **potential function** of $\mathbf{F}$.

(1) The **line integral** of $\int_C \mathbf{F} \cdot \mathbf{T} ds$ is **independent of path** in the region D if and only if $\mathbf{F} = \nabla f$ in D for some function f.

(2) Suppose that a vector function $\mathbf{F} = (P(x, y), Q(x, y))$ has continuous second order of derivatives in a region D. Then there is a function f in D such that $\nabla f = \mathbf{F}$ if and only if $P_y = Q_x$.

(3) Suppose $\mathbf{F} = (P(x, y), Q(x, y))$ and $P_y = Q_x$. Then the following steps will lead to the discovery of a potential function $f(x, y)$. (Another way to find a potential can be found in Example (3) below).
(Step 1) $f(x, y) = \int P(x, y)dx = f_1(x, y) + c(y)$, where $c(y)$ is an unknown function of y (viewed as constant for x).
(Step 2) Use the fact that $f_y = Q$ to get

$$Q = \frac{\partial f_1}{\partial y} + c'(y)$$

which results in a differential equation.
(Step 3) Solve this equation for $c(y)$, and substitute it back in Step 1 to get $f(x, y)$.

Example (1) Given $\mathbf{F} = (4x^2y - 5y^4, x^3 - 20xy^3)$, check if this is a conservative vector field, and if yes, find a potential function.

Solution: Here $P = 4x^2y - 5y^4$ and $Q = x^3 - 20xy^3$. First compute $P_y = 4x^2 - 20y^3$, and $Q_x = 3x^2 - 20y^3$. As $P_y \neq Q_x$. This is not conservative.

Example (2) Given $\mathbf{F} = (1 + ye^{xy}, 2y + xe^{xy})$, check if this is a conservative vector field, and if yes, find a potential function.

Solution: Here $P = 1+ye^{xy}$ and $Q = 2y+xe^{xy}$. First compute $P_y = e^{xy}+xye^{xy}$, and $Q_x = e^{xy}+xye^{xy}$. As $P_y = Q_x$, this is a conservative field.

To find a potential f, we first compute

$$f(x, y) = \int (1 + ye^{xy})dx = x + e^{xy} + c(y).$$

Then, set $2y + xe^{xy} = f_y = 0 + xe^{xy} + c'(y)$, to get $c'(y) = 2y$. Therefore, $c(y) = y^2$. Substitute $c(y) = y^2$ to $f(x, y)$ above to get

$$f(x, y) == x + e^{xy} + y^2.$$

Example (3) Given $\mathbf{F} = (2xy^2 + 3x^2, 2x^2y + 4y^3)$, check if this is a conservative vector field, and if yes, find a potential function.

Solution: Here $P = 2xy^2 + 3x^2$ and $Q = 2x^2y + 4y^3$. First compute $P_y = 4xy$, and $Q_x = 4xy$. As $P_y = Q_x$, this is a conservative field.

To find a potential function f, we note that the integral $\int_C Pdx + Qdy$ is independent of the path. Therefore, we note that one of the potential functions $f(x, y)$ satisfies $f(0, 0) = 0$, and so for fixed unknown (x_0, y_0), we choose C to be the straight line from$(0, 0)$ to (x_0, y_0). Thus the parametric equations of C are $x = tx_0$ and $y = ty_0$ with $0 \leq t \leq 1$. Thus as $f(0, 0) = 0$,

$$\begin{aligned} f(x_0, y_0) &= f(x_0, y_0) - f(0, 0) = \int_C Pdx + Qdy \\ &= \int_0^1 (2x_0y_0^2t^3 + 3x_0^2t^2)x_0dt + (2x_0^2y_0t^3 + 4y_0^3t^3)y_0dt \\ &= x_0^2y_0^2 + x_0^3 + y_0^4. \end{aligned}$$

Therefore, $f(x, y) = x^2y^2 + x^3 + y^4$.

Example (4) Show that the involved vector field is conservative and for some curve C on the xy-plane, evaluate the line integral

$$\int_{(0,0)}^{(1,3)} (2x - 3y)dx + (2y - 3x)dy$$

Solution: Here $P = 2x - 3y$ and $Q = 2y - 3x$. Note that $P_y = -3 = Q_x$. Therefore, the vector field is conservative. We shall choose an easy to compute path C.

Let $C_1 : 0 \leq x \leq 1$ and $y = 0$; and $C_2 : 0 \leq y \leq 3$ and $x = 1$. Then $C = C_1 + C_2$ and so

$$\int_{(0,0)}^{(1,3)} (2x - 3y)dx + (2y - 3x)dy = \int_0^1 2xdx + \int_0^3 (2y - 3)dy = (1 - 0) + (0 - 0) = 1.$$

Example (5) Given a conservative vector field $\mathbf{F} = (2x - y - z, 2y - x, 2z - x)$, find a potential function.

Solution: Here $P = 2x - y - z, Q = 2y - x$ and $R = 2z - x$. We can apply the method in Example (3) to find a potential function.

One of the potential functions $f(x, y, z)$ satisfies $f(0, 0, 0) = 0$, and so for fixed unknown (x_0, y_0, z_0), we choose C to be the straight line from$(0, 0, 0)$ to (x_0, y_0, z_0). Thus the parametric equations of C are $x = tx_0$, $y = ty_0$ and $z = tz_0$ with $0 \leq t \leq 1$. Thus as $f(0, 0, 0) = 0$,

$$\begin{aligned} f(x_0, y_0, z_0) &= f(x_0, y_0, z_0) - f(0, 0, 0) \\ &= \int_0^1 (2x_0 - y_0 - z^0)tx_0dt + (2y_0 - x_0)ty_0dt + (2z_0 - x_0)tz_0dt \\ &= x_0^2 - x_0y_0 - x_0z_0 + y_0^2 + z_0^2. \end{aligned}$$

Therefore, $f(x, y, z) = x^2 - xy - xz + y^2 + z^2$.