Determine if three points are coline

Example(1): Are the points $P(6,7,8)$, $Q(3,3,3)$ and $R(12,15,18)$ are on the same line?

Solution: Compute the vector $\mathbf{a} = \overline{QP} = (3,4,5)$ and $\mathbf{b} = \overline{QR} = (9,12,15)$. Note that $\mathbf{b} = 3\mathbf{a}$. Thus the two vectors are parallel, and so the points are on the same line.

Example(2): Are the points $P(0,-2,4)$, $Q(1,-3,5)$ and $R(4,-5,8)$ are on the same line?

Solution: Compute the vector $\mathbf{a} = \overline{PQ} = (1,-1,1)$ and $\mathbf{b} = \overline{PR} = (4,-3,4)$. Note that $\mathbf{b}$ cannot be a scalar product of $\mathbf{a}$. Thus the two vectors are not parallel, and so the points are not on the same line.

Compute the direction cosines of a vector

Example: Find the direction cosines of a vector represented by $\overline{PQ}$, where $P(2,-3,5)$ and $Q(1,0,-1)$ are two points.

Solution: Compute $\mathbf{a} = \overline{PQ} = (-1,3,-6)$. Then $|\mathbf{a}| = \sqrt{1+9+36} = \sqrt{46}$. Thus

$$\cos\alpha = \frac{-1}{\sqrt{46}}, \cos\beta = \frac{3}{\sqrt{46}}, \cos\gamma = \frac{-6}{\sqrt{46}}.$$

Verify the triangle inequality

Example: Given two vectors $\mathbf{a}$ and $\mathbf{b}$. Prove that $|\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$.

Solution: Since $|\cos\theta| \le 1$, we have $|\mathbf{a}\cdot\mathbf{b}| \le |\mathbf{a}|\cdot|\mathbf{b}|$. Thus

$$\begin{aligned}(|\mathbf{a}+\mathbf{b}|)^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = (\mathbf{a})^2 + 2\mathbf{a}\cdot\mathbf{b} + (\mathbf{b})^2 \\ &\le |\mathbf{a}|^2 + 2|\mathbf{a}|\cdot|\mathbf{b}| + |\mathbf{b}|^2 = (|\mathbf{a}|+|\mathbf{b}|)^2.\end{aligned}$$

Take square root both sides to get the desired inequality.