Compute iterated triple integrals

Useful facts: Suppose that $f(x, y, z)$ is continuous on a spacial region T, and T is z-**simple**: each line parallel to the z-axis intersects T (if not empty) in a a single line segment. An example of such region is

$$z_1(x, y) \leq z \leq z_2(x, y), \text{ and } (x, y) \text{ in } R,$$

where R is the vertical projection of T into the plane $z = 0$. Then

$$\int\int\int_T f(x,y,z)dV = \int\int_R \left(\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz\right) dA.$$

Example (1) Compute the value of the triple integral $\int\int\int_T f(x,y,z)dV$, where $f(x, y, z) = xy\sin z$, and T is the cube $0 \leq x \leq \pi$, $0 \leq y \leq \pi$ and $0 \leq z \leq \pi$.

Solution: Then R is the region $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$ on $z = 0$. Therefore,

$$\int\int\int_T xy\sin z dV = \int\int_R \left(\int_0^{\pi} xy\sin z dz\right) dA = \int_0^{\pi} x dx \int_0^{\pi} y dx \int_0^{\pi} \sin z dz = \frac{\pi^2}{2}\frac{\pi^2}{2}(2) = \frac{\pi^4}{2}.$$

Example (2) Compute the value of the triple integral $\int\int\int_T f(x,y,z)dV$, where $f(x, y, z) = 2x + 3y$, and T is the tetrahedron bounded by the coordinate planes and the first octant part of the plane with equation $2x + 3y + z = 6$.

Solution: Then R is the region on $z = 0$ bounded by $x = 0$, $y = 0$ and $2x + 3y = 6$, which can be described as $0 \leq x \leq 3$ and $0 \leq y \leq 2 - \frac{2x}{3}$. Note that here $z_1(x, y) = 0$ and $z_2(x, y) = 6 - 2x - 3y$. Therefore,

$$\begin{aligned}\int\int\int_T (2x+3y)dV &= \int\int_R \left(\int_0^{6-2x-3y} (2x+3y)dz\right) dA = \int_0^3 \int_0^{2-\frac{2x}{3}} \int_0^{6-2x-3y} (2x+3y)dzdydx \\ &= \int_0^3 \int_0^{2-\frac{2x}{3}} (12x + 18y - 4x^2 - 12xy - 9y^2)dydx = 18.\end{aligned}$$

Example (3) Compute the value of the triple integral $\int\int\int_T f(x,y,z)dV$, where $f(x, y, z) = x + y$, and T is the region between the surfaces $z = 2 - x^2$ and $z = x^2$ from $0 \leq y \leq 3$.

Solution: The two surfaces $z = 2 - x^2$ and $z = x^2$ intersect in two lines $x = 1$ and $x = -1$ (solve the equations $z = 2 - x^2$ and $z = x^2$ for x to find it out). Then R is the region on $z = 0$ bounded by the lines $x = -1$, $x = 1$, $y = 0$ and $y = 3$. Note that here $z_1(x, y) = x^2$ and $z_2(x, y) = 2 - x^2$. Therefore,

$$\int\int\int_T (x+y)dV = \int\int_R \left(\int_{x^2}^{2-x^2} (x+y)dz\right) dA = \int_0^3 \int_{-1}^1 \int_{x^2}^{2-x^2} (x+y)dzdxdy$$

Example (4) Find the volume of the solid bounded by these surfaces $z = x^2 + y^2, z = 0, x = 0, y = 0$ and $x + y = 1$.

Solution: The region R on the xy-plane is bounded by $x = 0, y = 0$ and $x + y = 1$. The surfaces $z = x^2 + y^2$ and $z = 0$ tops and bottoms the solid. Therefore,

$$\text{Volume } = \int\int_R \left(\int_0^{x^2+y^2} dz\right) dA = \int_0^1 \int_0^{1-x} (x^2 + y^2) dydx = \frac{1}{6}.$$

Compute the mass and the centroid of a solid

Useful facts: Suppose that $\delta(x, y, z)$ denotes the density function of a solid T. Let m denote the **mass** of T, and $\overline{x}, \overline{y}, \overline{z})$ denote the coordinates the **centroid** of T. and I_x, I_y and I_z denote the **moments** of T around the x-axis, the y-axis and the z-axis, respectively. Then

$$\begin{aligned}
m &= \int\int\int_T \delta(x, y, z)dV. \\
\overline{x} &= \frac{1}{m}\int\int\int_T x\delta(x, y, z)dV. \\
\overline{y} &= \frac{1}{m}\int\int\int_T y\delta(x, y, z)dV. \\
\overline{z} &= \frac{1}{m}\int\int\int_T z\delta(x, y, z)dV. \\
I_x &= \int\int\int_T (y^2 + z^2)\delta(x, y, z)dV. \\
I_y &= \int\int\int_T (x^2 + z^2)\delta(x, y, z)dV. \\
I_z &= \int\int\int_T (y^2 + x^2)\delta(x, y, z)dV.
\end{aligned}$$

Example (1) Find the centroid of the solid T bounded by $z = x^2$, $y + z = 4$, $y = 0$ and $z = 0$, given the density function $\delta \equiv 1$.

Solution: As the density is 1, this is the same as to find the volume. View the y-axis as the vertical axis. Then T lies between $y = 0$ and $y = 4 - z$. The region R on the xz-plane is bounded by $z = x^2$ and $z = 4$ (obtained by substituting $y = 0$ in $y + z = 4$). Therefore, the mass is

$$\begin{aligned}
m &= \int\int_R \left(\int_0^{4-z} dy\right) dA = \int_{-2}^2 \int_{x^2}^4 (4 - z)dzdx = \int_{-2}^2 \left[4z - \frac{z^2}{2}\right]_{x^2}^4 dx \\
&= \int_{-2}^2 \left(8 - 4x^2 + \frac{x^4}{2}\right) dx = \left[8x - \frac{4x^3}{3} + \frac{x^5}{10}\right]_{-2}^2 = 64\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{10}\right) = \frac{256}{15}.
\end{aligned}$$

The coordinates of the centroid is

$$\overline{x} = \frac{15}{256}\int_{-2}^2 \int_{x^2}^4 \int_0^{4-z} xdydzdx = \frac{15}{256}\int_{-2}^2 \int_0^{x^2} (4 - z)xdzdx = \int_{-2}^2 \left(8x - 4x^3 + \frac{x^5}{2}\right) dx = 0$$

$$
\begin{aligned}
\overline{y} &= \frac{15}{256}\int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{4-z} y\,dydzdx = \frac{15}{256}\int_{-2}^{2}\int_{x^2}^{4}\frac{(4-z)^2}{2}dzdx = \frac{15}{256}\int_{-2}^{2}\frac{1}{2}\int_{x^2}^{4}(16-8z+z^2)dzdx \\
&= \frac{15}{512}\int_{-2}^{2}\left[16z-4z^2+\frac{z^3}{3}\right]_{x^2}^{4}dx = \frac{15}{512}\int_{-2}^{2}\left[\frac{64}{3}-16x^2+4x^4-\frac{x^6}{3}\right]dx \\
&= \frac{15}{512}\left[\frac{64x}{3}-\frac{16x^3}{3}+\frac{4x^5}{5}-\frac{x^7}{213}\right]_{-2}^{2} = \frac{15}{512}\left[\frac{256}{3}-\frac{256}{3}+\frac{256}{5}-\frac{256}{21}\right] = \frac{15}{2}\left(\frac{1}{5}-\frac{1}{7}\right) = \frac{8}{7}. \\
\overline{z} &= \frac{15}{256}\int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{4-z} z\,dydzdx = \frac{15}{256}\int_{-2}^{2}\int_{x^2}^{4}(4z-z^2)dzdx = \frac{15}{256}\int_{-2}^{2}\left[2z-\frac{z^3}{3}\right]_{x^2}^{4}dx \\
&= \frac{15}{256}\int_{-2}^{2}\left[\frac{32}{3}-2x^4+\frac{x^6}{3}\right]dx = \frac{15}{256}\int_{-2}^{2}\left[\frac{32x}{3}-\frac{2x^5}{5}+\frac{x^7}{21}\right]_{-2}^{2} \\
&= \frac{15}{256}\left[\frac{128}{3}-\frac{128}{5}+\frac{256}{21}\right] = \frac{12}{7}.
\end{aligned}
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