Compute double integrals in polar coordinates

Useful facts: Suppose that $f(x, y)$ is continuous on a region R in the plane $z = 0$.
(1) If the region R is bounded by $\alpha \le \theta \le \beta$ and $a \le r \le b$, then

$$\int\int_R f(x, y)dA = \int_\alpha^{beta} \int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta.$$

(2) If the region R is bounded by $\alpha \le \theta \le \beta$ and $r_1(\theta) \le r \le r_2(\theta)$ (called a **radially simple** region), then

$$\int\int_R f(x, y)dA = \int_\alpha^{beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta.$$

Example (1) Find the volume of a sphere of radius a by double integration.

Solution: We can view that the center of the sphere is at the origin $(0, 0, 0)$, and so the equation of the sphere is $x^2 + y^2 + z^2 = a^2$. We then can compute the volume of the upper half part of the sphere and multiply our answer by 2.

$$V = 2\int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}dxdy.$$

To compute this integral, we observe that the polar coordinates may be a better mechanism in this case. With polar coordinates, the function $z = \sqrt{a^2 - x^2 - y^2}$ becomes $z = \sqrt{a^2 - r^2}$, over the region $-\pi \le \theta\pi$ and $0 \le r \le a$. Therefore, using polar coordinates, we have (using $u = a^2 - r^2$ and $2rdr = -du$ to start with)

$$V = 2\int_{-\pi}^{\pi} \int_0^a \sqrt{a^2 - r^2}rdrd\theta = \int_{-\pi}^{\pi} \int_0^{a^2} u^{1/2}dud\theta = 2\pi\frac{2a^3}{3} = \frac{4\pi a^3}{3}.$$

Example (2) Find the area of the region R bounded by one loop of $r = 2\cos 2\theta$.

Solution: In the interval $[-\pi, \pi]$ of θ, $\cos 2\theta = 0$ exactly at $\theta = \pm\frac{\pi}{4}$ and $\theta = \pm\frac{3\pi}{4}$. For one loop, this is the case when $\alpha = -\frac{\pi}{4}$ and $\beta = \frac{\pi}{4}$, while $r_1 = 0$ and $r_2 = 2\cos 2\theta$. Use the fact that $\sin\frac{\pm\pi}{2} = \pm 1$ to get

$$A = \int\int_R dA = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{2\cos 2\theta} rdrd\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\cos^2 2\theta d\theta = \frac{\pi}{2}.$$

Example (3) Find the area of the region R inside the smaller loop of $r = 1 - 2\sin\theta$.

Solution: In the interval $[-\pi, \pi]$ of θ, $\sin\theta = \frac{1}{2}$ exactly at $\theta = \pm\frac{\pi}{4}$ and $\theta = \pm\frac{3\pi}{4}$. For the smaller loop, this is the case when $\alpha = \frac{\pi}{4}$ and $\beta = \frac{3\pi}{4}$, while $r_1 = 0$ and $r_2 = 1 - 2\sin\theta$. Thus

$$A = \int\int_R dA = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{1-2\sin\theta} rdrd\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{(1 - 2\sin\theta)^2}{2}d\theta = \frac{2\pi - 3\sqrt{3}}{2}.$$

Example (4) Find the volume of the solid that lies below the surface $z = x^2 + y^2$ over the region R bounded by $r = 2\cos\theta$.

Solution: In the interval $[-\pi, \pi]$ of θ, $\cos\theta = 0$ exactly at $\theta = \pm\frac{\pi}{2}$. This is the case when $\alpha = -\frac{\pi}{2}$ and $\beta = \frac{\pi}{2}$, while $r_1 = 0$ and $r_2 = 2\cos\theta$. Thus

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^3 dr d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(2\cos\theta)^4}{4} d\theta = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta)^2}{4}\theta = \frac{3\pi}{2}.$$

Example (5) Evaluate the double integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{4-x^2-y^2}} dy dx.$$

Solution: Change to polar coordinates. Then

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{4-x^2-y^2}} dy dx = \int_0^{\frac{\pi}{2}} \int_0^1 \frac{1}{4-r^2} r dr d\theta = \frac{\pi}{2} \int_0^1 \frac{1}{4-r^2} r dr = \frac{\pi(2-\sqrt{3})}{2}.$$

Example (6) Find the volume of the solid that lies below the surface $z = 1 + x$ and above the plane $z = 0$ over the region R bounded by $r = 1 + \cos\theta$.

Solution: In the interval $[-\pi, \pi]$ of θ, $\cos\theta = -1$ exactly at $\theta = \pm\pi$. This is the case when $\alpha = -\pi$ and $\beta = \pi$, while $r_1 = 0$ and $r_2 = 1 + \cos\theta$. Thus

$$\begin{aligned} V &= \int_{-\pi}^{\pi} \int_0^{1+\cos\theta} (1 + r\cos\theta) r dr d\theta = \int_{-\pi}^{\pi} \left[\frac{(1+\cos\theta)^2}{2} + \frac{(1+\cos\theta)^3}{3} \cos\theta \right] d\theta \\ &= \frac{1}{6} \int_{-\pi}^{\pi} (3 + 9\cos^2\theta + 2\cos^4\theta) d\theta = \frac{11}{4}\pi. \end{aligned}$$

Example (7) Find the volume of the solid bounded by the paraboloid $z = 12 - 2x^2 - y^2$ and $z = x^2 + 2y^2$.

Solution: The intersection of the two surfaces , when projected down to the $z = 0$ plane, is the common solution of both $z = 12 - 2x^2 - y^2$ and $z = x^2 + 2y^2$, which is a curve with equation $3x^2 + 3y^2 = 14$, or $x^2 + y^2 = 4$ on the plane $z = 0$. In terms of polar coordinates, the region R bounded by this curve (a circle centered at the origin with radius 2) is also bounded by $-\pi \le \theta \le \pi$ and $0 \le r \le 2$. The top surface is $z = 12 - 2x^2 - y^2$ and the bottom one is $z = x^2 + 2y^2$. Thus

$$\begin{aligned} V &= \int_{-\pi}^{\pi} \int_0^2 (12 - 2x^2 - y^2 - x^2 - 2y^2) r dr d\theta = 3\int_{-\pi}^{\pi} \int_0^2 (4 - x^2 - y^2) r dr d\theta \\ &= 3\int_{-\pi}^{\pi} \int_0^2 (4 - r^2) r dr d\theta = 6\pi \left[2r^2 - \frac{r^4}{4} \right]_0^2 = 24\pi. \end{aligned}$$