Evaluate double integrals over general regions

Useful facts: Suppose that $f(x, y)$ is continuous on a region R.
(1) If R can be described by $a \le x \le b, y_1(x) \le y \le y_2(x)$, (that is, R is **vertically simple**), then

$$\int\int_R f(x,y)dA = \int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x,y)dy \right) dx.$$

(2) If R can be described by $c \le y \le d, x_1(y) \le x \le x_2(y)$, (that is, R is **horizontally simple**), then

$$\int\int_R f(x,y)dA = \int_c^d \left(\int_{x_1(y)}^{x_2(y)} f(x,y)dx \right) dy.$$

Example (1) Evaluate

$$\int_0^1 \int_y^{\sqrt{y}} (x+y)dxdy.$$

Solution: Convert the double integral into iterated integrals:

$$\int_0^1 \int_y^{\sqrt{y}} (x+y)dxdy = \int_0^1 \left(\frac{x^2}{2} + yx \right)_y^{\sqrt{y}} dy = \int_0^1 \left(\frac{y}{2} + y^{3/2} - \frac{y^2}{2} - y^2 \right) dy = \left[\frac{y^2}{4} + \frac{2y^{5/2}}{5} - \frac{y^3}{2} \right]_0^1 = \frac{3}{20}.$$

Example (2) Evaluate the integral of the function $f(x, y) = x^2$ over the region R, which is bounded by the parabola $y = 2 - x^2$ and the line $y = -4$.

Solution: Note that the two curves $y = 2 - x^2$ and $y = -4$ intersect at $(\sqrt{6}, -4)$ and $(-\sqrt{6}, -4)$ (this can be obtained by solving the system of equations $y = 2 - x^2$ and $y = -4$ simultaneously).

Then use it to set up the double integral and evaluate it:

$$\int_{-\sqrt{6}}^{\sqrt{6}} \int_{-4}^{2-x^2} x^2 dxdy = \int_{-\sqrt{6}}^{\sqrt{6}} (6x^2 - x^4)dx = \left[2x^3 - \frac{x^5}{5} \right]_{-\sqrt{6}}^{\sqrt{6}} = \frac{48}{5}\sqrt{6}.$$

Example (3) Evaluate the integral of the function $f(x, y) = \sin x$ over the region R, which is bounded by the x-axis, and the curve $y = \cos x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

Solution: As the x bounds are explicitly given, it may be easier to view the region as a vertically simple one.

Set up the double integral and evaluate it:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos x} \sin x dydx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \cos x dx = \left[\frac{\sin^2 x}{2} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0.$$

Example (4) Evaluate the integral

$$\int_0^1 \int_y^1 e^{-x^2} dx dy.$$

Solution: Direct computation encounters difficulty. Note that the region R is bounded by the lines $y = x$, $x = 1$ and $y = 0$. Change the order of integration to compute this integral (use a Calculus I substitution $u = x^2$ in the third equality).

$$\int_0^1 \int_y^1 e^{-x^2} dx dy = \int_0^1 \int_0^x e^{-x^2} dy dx = \int_0^1 e^{-x^2} x dx = \frac{1}{2} \int_0^1 e^{-u} du = -\frac{1}{2} e^{-u} \Big|_0^1 = \frac{e-1}{2e}.$$

Example (5) Evaluate the integral

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y} dy dx.$$

Solution: Direct computation encounters difficulty. Note that the region R is bounded by the lines $y = x$, $x = \pi$ and $y = 0$. Change the order of integration to compute this integral (note that $\cos 0 = 1$ and $\cos \pi = -1$ in the last step).

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y} dy dx = \int_0^\pi \int_0^y \frac{\sin y}{y} dx dy = \int_0^\pi \sin y dy = 2.$$

Example (6) Evaluate the integral

$$\int_0^1 \int_y^1 \frac{1}{1+x^4} dx dy.$$

Solution: Direct computation encounters difficulty. Note that the region R is bounded by the lines $y = x$, $x = 1$ and $y = 0$. Change the order of integration to compute this integral (use $u = x^2$ in the third equality and note that $\tan^{-1}(1) = \frac{\pi}{4}$ and $\tan^{-1}(0) = 0$).

$$\int_0^1 \int_y^1 \frac{1}{1+x^4} dx dy = \int_0^1 \int_0^x \frac{1}{1+x^4} dy dx = \int_0^1 \frac{x}{1+x^4} dx = \frac{1}{2} \int_0^1 \frac{1}{1+u^2} du = \frac{1}{2} \tan^{-1} u \Big|_0^1 = \frac{\pi}{4}.$$