Compute the (double integral) Riemann sum

Example (1) Approximate the integral

$$\int\int_R (4x^3 + 6xy^2)dA$$

over the rectangle $R = [1, 3] \times [-2, 1]$ by partitioning R into six unit squares $R_1, \cdots, R_6$ and by selecting each (x_i^*, y_i^*) as the lower left corner of the rectangle R_j.

Solution: As the lower left corners of these unit squares are at $(1, -2), (2, -2), (1, -1), (2, -1), (1, 0)$ and $(2, 0)$, Therefore, the corresponding Riemann is

$$\begin{aligned}\sum_{i=1}^{6} f(x_i^*, y_i^*)\Delta A_i &= f(1, -2)(1) + f(2, -2)(1) + f(1, -1)(1) + f(2, -1)(1) + f(1, 0)(1) + f(2, 0)(1) \\ &= 28 + 80 + 10 + 44 + 4 + 32 = 198.\end{aligned}$$

Example (2) Approximate the integral

$$\int\int_R (x^2 + y^2)dA$$

over the rectangle $R = [0, 2] \times [0, 3]$ by partitioning R into six unit squares $R_1, \cdots, R_6$ and by selecting each (x_i^*, y_i^*) as the upper right corner of the rectangle R_j.

Solution: As the upper right corners of these unit squares are at $(1, 1), (2, 1), (1, 2), (2, 2), (1, 3)$ and $(2, 3)$, Therefore, the corresponding Riemann is

$$\begin{aligned}\sum_{i=1}^{6} f(x_i^*, y_i^*)\Delta A_i &= f(1, 1)(1) + f(2, 1)(1) + f(1, 2)(1) + f(2, 2)(1) + f(1, 3)(1) + f(2, 3)(1) \\ &= 2 + 5 + 5 + 8 + 10 + 13 = 43.\end{aligned}$$

Evaluate double integrals over rectangular regions

Useful facts: If $f(x, y)$ is continuous on the square region $R = [a, b] \times [c, d]$, then

$$\int_a^b \int_c^d f(x, y)dydx = \int_a^b \left(\int_c^d f(x, y)dy \right) dx.$$

Example (1) Evaluate

$$\int_{-2}^{1} \int_{2}^{4} x^2y^3dydx.$$

Solution: Convert the double integral into iterated integrals:

$$\int_{-2}^{1}\int_{2}^{4} x^2y^3dydx = \int_{-2}^{1} x^2dx \int_{2}^{4} y^3dy = \left[\frac{x^3}{3}\right]_{-2}^{1}\left[\frac{y^4}{4}\right]_{2}^{4} = \left(\frac{1}{3}-\frac{-8}{3}\right)\left(\frac{4^4}{4}-\frac{2^4}{4}\right) = (3)(60) = 180.$$

Example (2) Evaluate

$$\int_{1}^{3}\int_{-3}^{-1}(x^3y - xy^3)dydx.$$

Solution: Convert the double integral into iterated integrals:

$$\begin{aligned}
\int_{1}^{3}\int_{-3}^{-1}(x^3y - xy^3)dydx &= \int_{1}^{3}\int_{-3}^{-1} x^3ydydx - \int_{1}^{3}\int_{-3}^{-1} xy^3dydx \\
&= \int_{1}^{3} x^3dx \int_{-3}^{-1} ydy - \int_{1}^{3} xdx \int_{-3}^{-1} y^3dy \\
&= \left[\frac{x^4}{4}\right]_{1}^{3}\left[\frac{y^2}{2}\right]_{-3}^{-1} - \left[\frac{x^2}{2}\right]_{1}^{3}\left[\frac{y^4}{4}\right]_{-3}^{-1} \\
&= \left(\frac{81}{4}-\frac{1}{4}\right)\left(\frac{1}{2}-\frac{9}{2}\right) - \left(\frac{9}{2}-\frac{1}{2}\right)\left(\frac{1}{4}-\frac{81}{4}\right) \\
&= (20)(-4) - (4)(-20) = 0.
\end{aligned}$$

Example (3) Evaluate

$$\int_{1}^{2}\int_{1}^{3}\left(\frac{x}{y}+\frac{y}{x}\right)dydx.$$

Solution: Convert the double integral into iterated integrals:

$$\begin{aligned}
\int_{1}^{2}\int_{1}^{3}\left(\frac{x}{y}+\frac{y}{x}\right)dydx &= \int_{1}^{2}\left(\int_{1}^{3}\left(\frac{x}{y}+\frac{y}{x}\right)dy\right)dx \\
&= \int_{1}^{2}\left(\left[x\ln|y| + \frac{y^2}{2x}\right]_{1}^{3}\right)dx \\
&= \int_{1}^{2}\left(x\ln 3 + \frac{9}{2x} - x\ln 1 - \frac{1}{2x}\right)dx \\
&= \left[\frac{x^2\ln 3}{2} + 4\ln|x|\right]_{1}^{2} = 2\ln 3 + 4\ln 2 - \frac{\ln 3}{2} - 0 \\
&= \frac{3\ln 3}{2} + 4\ln 2.
\end{aligned}$$