Find the maximum directional derivatives of a function at a given point

Fact: The the maximum directional derivatives of a function f at a given point P is obtained in the same direction of the gradient vector of f at P. Namely, it occurs at the direction of

$$\mathbf{u} = \frac{\nabla f}{|\nabla f|}, \text{ and so the maximum directional derivative of } f \text{ at } P \text{ is } |\nabla f|.$$

Example (1) : Find the gradient vector of $f(x, y) = 3x^2 - 5y^2$ at the point $P(2, -3)$.

Solution: First compute $\nabla f = (6x, -10y)$. At P, the answer is $\nabla f(2, -3) = (12, 30)$.

Example (2) : Find the gradient vector of $f(x, y) = (2x-3y+5z)^5$ at the point $P(-5, 1, 3)$.

Solution: First compute $\nabla f = (10(2x - 3y + 5z)^4, -15(2x - 3y + 5z)^4, 25(2x - 3y + 5z)^4)$. At P, the answer is $\nabla f(-5, 1, 3) = (160, -240, 400)$.

Find tangent planes with normal gradient vector

Fact: A normal vector of the plane tangent to the surface with equation $F(x, y, z) = 0$ at a given point $P(x_0, y_0, z_0)$ can be the gradient vector $\nabla F(P) = (F_x(P), F_y(P), F_z(P))$. Therefore an equation of this tangent plane is

$$F_x(P)(x - x_0) + F_y(P)(y - y_0) + F_z(P)(z - z_0) = 0.$$

Example (1) : Find an equation of the plane tangent to the surface $x^{1/3} + y^{1/3} + z^{1/3} = 1$ at $P(1, -1, 1)$.

Solution: Let $F(x, y, z) = x^{1/3}+y^{1/3}+z^{1/3}-1$. Then the surface has equation $F(x, y, z) = 0$. Compute $\nabla F = \frac{1}{3}(x^{-2/3}, y^{-2/3}, z^{-2/3})$. At P, $\nabla F(P) = \frac{1}{3}(1, 1, 1)$ and so an equation of the tangent plane is

$$(x - 1) + (y + 1) + (z - 1) = 0.$$

Example (2) : Find an equation of the plane tangent to the surface $xyz+x^2-2y^2+z^3 = 14$ at $P(5, -2, 3)$.

Solution: Let $F(x, y, z) = xyz + x^2 - 2y^2 + z^3 - 14$. Then the surface has equation $F(x, y, z) = 0$. Compute $\nabla F = (yz + 2x, xz - 4y, xy + 3z^2)$. At P, $\nabla F(P) = (4, 23, 17)$ and so an equation of the tangent plane is

$$4(x - 5) + 23(y + 2) + 17(z - 3) = 0.$$