Compute the Gradient Vector

Fact: The gradient vector of functions $g(x, y)$ and $f(x, y, z)$ are, respectively,

$$\nabla g = (g_x, g_y) \text{ and } \nabla f = (f_x, f_y, f_z).$$

Example (1) : Find the gradient vector of $f(x, y) = 3x^2 - 5y^2$ at the point $P(2, -3)$.

Solution: First compute $\nabla f = (6x, -10y)$. At P, the answer is $\nabla f(2, -3) = (12, 30)$.

Example (2) : Find the gradient vector of $f(x, y) = (2x-3y+5z)^5$ at the point $P(-5, 1, 3)$.

Solution: First compute $\nabla f = (10(2x - 3y + 5z)^4, -15(2x - 3y + 5z)^4, 25(2x - 3y + 5z)^4)$. At P, the answer is $\nabla f(-5, 1, 3) = (160, -240, 400)$.

Compute the directional derivative of a function f in the direction v

Fact: The directional derivative of a function f in the direction $\mathbf{v}$ is the dot product $\nabla f \cdot \frac{\mathbf{v}}{|\mathbf{v}|}$.

Example (1) : Find the directional derivative of $f(x, y) = e^x \sin y$ at the point $P(0, \frac{\pi}{4})$ in the direction $\mathbf{v} = (1, -1)$.

Solution:
(Step 1) First compute $\nabla f = (e^x \sin y, e^x \cos y)$. At P, $\nabla f(0, \frac{\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.
(Step 2) Find the unit vector that has the same direction as $\mathbf{v}$. Compute $|\mathbf{v}| = \sqrt{2}$. Then $\mathbf{u} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.
(Step 3) Find the answer. Compute the dot product $\nabla f(0, \frac{\pi}{4}) \cdot \mathbf{u} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \cdot (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = 0$.

Example (2) : Find the directional derivative of $f(x, y, z) = \ln(1 + x^2 + y^2 + z^2)$ at the point $P(1, -1, 1)$ in the direction $\mathbf{v} = (2, -2, -3)$.

Solution:
(Step 1) First compute

$$\nabla f = \left(\frac{2x}{1 + x^2 + y^2 + z^2}, \frac{2y}{1 + x^2 + y^2 + z^2}, \frac{2z}{1 + x^2 + y^2 + z^2} \right).$$

At P, $\nabla f(1, -1, 1) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.
(Step 2) Find the unit vector that has the same direction as $\mathbf{v}$. Compute $|\mathbf{v}| = \sqrt{4 + 4 + 9} = \sqrt{17}$. Then $\mathbf{u} = (\frac{2}{\sqrt{17}}, -\frac{2}{\sqrt{17}}, -\frac{3}{\sqrt{17}})$.
(Step 3) Find the answer. Compute the dot product $\nabla f(1, -1, 1) \cdot \mathbf{u} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \cdot (\frac{2}{\sqrt{17}}, -\frac{2}{\sqrt{17}}, -\frac{3}{\sqrt{17}}) = -\frac{3}{2\sqrt{17}}$.