Find the maximum and minimum values of $f(x, y)$ over a region R

Key Idea: The maximum and the minimum mast occur at either a critical point or on the boundary of the region.

Example (1) : Find the maximum and minimum values of $f(x, y) = x^2 + y^2 - x$ over a region R, which is the square with vertices at $(\pm 1, \pm 1)$.

Solution: Compute $f_x = 2x - 1$ and $f_y = 2y$, and so $(1/2, 0)$ is the only critical point inside R. The value of $f(1/2, 0) = -\frac{1}{4}$.

The region of R consists of 4 line segments: $L_1 : (x, 1)$ with $-1 \leq x \leq 1$, $L_2 : (x, -1)$ with $-1 \leq x \leq 1$,$L_3 : (1, y)$ with $-1 \leq y \leq 1$, $L_4 : (-1, y)$ with $-1 \leq y \leq 1$.

Compute the extremal values of $f(x, y)$ on each segment of boundary as follows.

On L_1, $f(x, 1) = x^2 + 1 - x$ with x in the closed interval $[-1, 1]$. Using Calculus I technique, the maximum is $f(-1, 1) = 3$ and the minimum is $f(\frac{1}{2}, 1) = \frac{3}{4}$.

On L_2, $f(x, -1) = x^2 + 1 - x$ with x in the closed interval $[-1, 1]$. Using Calculus I technique, the maximum is $f(-1, 1) = 3$ and the minimum is $f(\frac{1}{2}, 1) = \frac{3}{4}$.

On L_3, $f(1, y) = y^1$ with y in the closed interval $[-1, 1]$. Using Calculus I technique, the maximum is $f(1, 1) = 1$ and the minimum is $f(1, 0) = 0$.

On L_4, $f(-1, y) = 2 + y^1$ with y in the closed interval $[-1, 1]$. Using Calculus I technique, the maximum is $f(-1, 1) = 3$ and the minimum is $f(-1, 0) = 2$.

Compare all the values computed thus far, we conclude that the maximum attained by f in R is $f(-1, 1) = 3$ and the minimum attained by f in R is $f(1/2, 0) = -\frac{1}{4}$.

Example (2) : Find the maximum and minimum values of $f(x, y) = x^2 + y^2 - x - y$ over a region R, which is the triangular region with vertices at $(0, 0)$, $(2, 0)$ and $(0, 2)$.

Solution: Compute $f_x = 2x - 1$ and $f_y = 2y - 1$, and so $(1/2, 1/2)$ is the only critical point inside R. The value of $f(1/2, 1/2) = -\frac{1}{2}$.

The region of R consists of 3 line segments: $L_1 : (x, 0)$ with $0 \leq x \leq 2$, $L_2 : (0, y)$ with $0 \leq y \leq 2$,$L_3 : (x, 2 - x)$ with $0 \leq x \leq 2$.

Compute the extremal values of $f(x, y)$ on each segment of boundary as follows.

On L_1, $f(x, 0) = x^2 - x$ with x in the closed interval $[0, 2]$. Using Calculus I technique, the maximum is $f(0, 2) = 2$ and the minimum is $f(\frac{1}{2}, 0) = -\frac{1}{4}$.

On L_2, $f(0, y) = y^2 - y$ with y in the closed interval $[0, 2]$. Using Calculus I technique, the maximum is $f(0, 2) = 2$ and the minimum is $f(0, \frac{1}{2}) = -\frac{1}{4}$.

On L_3, $f(x, 2 - x) = x^2 + (2 - x)^2 - x - (2 - x) = 2x^2 - 4x + 2$ with x in the closed interval $[0, 2]$. Using Calculus I technique, the maximum is $f(0, 2) = f(2, 0) = 2$ and the minimum is $f(1, 1) = 0$.

Compare all the values computed thus far, we conclude that the maximum attained by f in R is $f(0, 2) = f(2, 0) = 2$ and the minimum attained by f in R is $f(1/2, 1/2) = -\frac{1}{2}$.