Use limit laws to evaluate limits

Example (1) : Compute $\lim_{(x,y)\to(0,0)} exp\left(\frac{1}{x^2+y^2}\right)$.

Solution: Set $r^2 = x^2 + y^2$. Then

$$\begin{aligned}\lim_{(x,y)\to(0,0)} exp\left(\frac{1}{x^2+y^2}\right) &= \lim_{r\to 0} e^{\frac{-1}{r^2}} = \lim_{r\to 0} \frac{1}{e^{\frac{1}{r^2}}} \qquad \text{set } u = \frac{1}{r^2} \\ &= \lim_{u\to\infty} \frac{1}{e^u} = 0\end{aligned}$$

Example (2) : Compute $\lim_{(x,y,z)\to(1,1,0)} \frac{xy-z}{\cos(xyz)}$.

Solution: As $\lim_{(x,y,z)\to(1,1,0)} xy - z = 1$ and $\lim_{(x,y,z)\to(1,1,0)} \cos(xyz) = \cos 0 = 1$, the answer is 1.

Determine if the limit does not exist

Example (1) : Show that the limit $\lim_{(x,y)\to(2,-2)} \frac{4-xy}{4+xy}$.

Solution: As the denominator approaches to zero but the numerator is not, the limit does not exist.

Example (2) : Show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$.

Solution: Let (x, y) approach to $(0, 0)$ along the straight line $y = mx$, where m can take any real value. This amounts to substitute $y = mx$ in the limit and so $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(1-m^2)x^2}{(1+m^2)x^2} = \frac{1-m^2}{1+m^2}$. As m can be 1 or 2, the same limit, when (x, y) goes to $(0, 0)$ along $y = x$, is 0; and when (x, y) goes to $(0, 0)$ along $y = 2x$, is not zero. Therefore, the limit does not exist.

Example (3) : Determine if the limit $\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$ exists or not.

Solution: Convert to spherical coordinates to get

$$\begin{aligned}\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} &= \lim_{\rho\to 0} \frac{\rho^3 \sin\phi\cos\theta\sin\phi\sin\theta\cos\phi}{\rho^2} \\ &= \lim_{\rho\to 0} \rho\sin\phi\cos\theta\sin\phi\sin\theta\cos\phi = 0,\end{aligned}$$

as for any value of θ and ϕ, we always have

$$|\sin\phi\cos\theta\sin\phi\sin\theta\cos\phi| \leq 1.$$

Therefore, the limit exists and its value is 0.