Convert equations from one coordinate system to another: I

Useful Facts

Cylindrical	Rectangle	Spherical	Rectangle
$\begin{cases} r^2 &= x^2+y^2 \\ \theta &= \tan^{-1}\frac{y}{x} \\ z &= z \end{cases}$	$\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{cases}$	$\begin{cases} \rho^2 &= x^2+y^2+z^2 \\ \phi &= \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} \\ \theta &= \tan^{-1}\frac{y}{x} \end{cases}$	$\begin{cases} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{cases}$

Example (1) : Describe the graph $r = 5$ in cylindrical coordinates.

Solution: As z and θ can take any values, the graph of $r = 5$ is an infinite cylinder with z-axis as the axis of the cylinder, and every point on the graph has distance 5 to the z-axis. Notice that the straight line $L : x = \sqrt{5}, y = 0, z = t$ is on this graph, this graph can also be obtained by rotating L about the z-axis.

Example (2) : Describe the graph $\theta = \frac{\pi}{4}$ in cylindrical (or spherical) coordinates.

Solution: As z and r can take any values, the graph of $\theta = \frac{\pi}{4}$ consists of all the points in the space whose θ value is $\frac{\pi}{4}$, and so it is a plane that contains the z-axis.

One can also convert this equation into rectangular coordinates by using the formula $y = x\tan\theta = x\tan\frac{\pi}{4} = x$. Therefore, we can equivalently describe the graph of $y = x$ in the space. Everybody knows that $y = x$ represents a plane in the space.

Remark As θ in cylindrical coordinates represents the same measure as in spherical coordinates, in this example, $\theta = \frac{\pi}{4}$ in spherical coordinates has the same graph $y = x$.

Example (3) : Describe the graph $\phi = \frac{\pi}{6}$ in spherical coordinates.

Solution: As ρ and θ can take any valid values, the graph of $\phi = \frac{\pi}{6}$ consists of all the points in the space whose ϕ value is $\frac{\pi}{6}$, and so it is a (two penning) cone with its vertex at the origin, and with its axis being the z-axis.

One can also use algebraic techniques to see what the graph is like. Apply the formula $r = z\tan\phi$ and $r^2 = x^2 + y^2$, and substitute ϕ by $\frac{\pi}{6}$ (knowing that $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$). This yields the equation for the graph

$$x^2 + y^2 = \frac{z^2}{3}.$$

Therefore, the graph can be obtained from the straight line $z = \sqrt{3}x$ on the xz-plane by rotating this line about the z-axis.