Write an equation for the surface generated by revolving a plane curve around an axis

Key idea: Use the fact that the distance from any point on a circle to the center is always equal to the distance from any point on a circle to the center.

Example (1): Write an equation for the surface generated by revolving the curve $4x^2 + 9y^2 = 36$ on the plane $z = 0$ around the y-axis.

Solution: Let $P(x, y, z)$ be a generic point on the surface of revolution. Fix a point $Q(x_1, y, 0)$ on the curve with the same y-coordinate as P. Then we have

$$4x_1^2 + 9y^2 = 36. \tag{1}$$

The square of the distance from Q to the y-axis is x_1^2, and the square of the distance from P to the y-axis is $x^2 + z^2$. As the two distances should be the same, we have $x_1^2 = x^2 + z^2$. Therefore, the wanted equation can be obtained by replacing x_1^2 by $x^2 + z^2$ in Equation (1):

$$4x^2 + 9y^2 + 4z^2 = 36.$$

Example (2): Write an equation for the surface generated by revolving the curve $x = 2z^2$ on the plane $y = 0$ around the x-axis.

Solution: Let $P(x, y, z)$ be a generic point on the surface of revolution. Fix a point $Q(x, 0, z_1)$ on the curve with the same x-coordinate as P. Then we have

$$x_1 = 2z^2. \tag{2}$$

The square of the distance from Q to the x-axis is z_1^2, and the square of the distance from P to the x-axis is $y^2 + z^2$. As the two distances should be the same, we have $z_1^2 = y^2 + z^2$. Therefore, the wanted equation can be obtained by replacing z_1^2 by $y^2 + z^2$ in Equation (2):

$$x = 4(y^2 + z^2).$$

Example (3): Write an equation for the surface generated by revolving the curve $(y - z)^2 + z^2 = 1$ on the plane $x = 0$ around the z-axis.

Solution: Let $P(x, y, z)$ be a generic point on the surface of revolution. Fix a point $Q(0, y_1, z)$ on the curve with the same z-coordinate as P. Then we have

$$(y_1 - z)^2 + z^2 = 1, \text{ which is the same as } y_1^2 - 2y_1 z + 2z^2 = 1. \tag{3}$$

The square of the distance from Q to the z-axis is y_1^2, and the square of the distance from P to the z-axis is $x^2 + y^2$. As the two distances should be the same, we have $y_1^2 = x^2 + y^2$. Accordingly, $y_1 = \pm\sqrt{x^2 + y^2}$. Therefore, the wanted equation can be obtained by replacing y_1 by $\pm\sqrt{x^2 + y^2}$ in Equation (3) (it has two branches):

$$x^2 + y^2 \pm 2z\sqrt{x^2 + y^2} + 2z^2 = 1.$$