Compute the arc length of curves

Example: Compute length of arc of the curve $x = t^2/2, y = \ln t, z = t\sqrt{2}$ from $t = 1$ to $t = 2$.

Solution: Apply the arc length formula

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

to get the answer

$$s = \int_1^2 \sqrt{t^2 + \frac{1}{t^2} + (\sqrt{2})^2} dt = \int_1^2 \sqrt{\frac{t^4 + 1 + 2t^2}{t^2}} dt = \int_1^2 \frac{t^2 + 1}{t} dt = \left[\frac{t^2}{2} + \ln t\right]_1^2 = 2 + \ln 2 - \frac{1}{2} = \frac{3}{2} + \ln 2.$$

Compute the curvature of a curve

Example Find the curvature of a plane curve $x = t - 1$ and

Solution: Apply the plane curve curvature formula

$$\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{[(x'(t))^2 + (y'(t))^2]^{3/2}}.$$

First compute $x'(t) = 1$ and $x''(t) = 0$, $y'(t) = 2t + 3$ and $y''(t) = 2$. Then compute $(x'(t))^2 + (y'(t))^2 = 1 + 4t^2 + 12t + 9 = 4t^2 + 12t + 10$. When $t = 2$, the curvature is

$$\kappa(2) = \frac{|2 - 0|}{[4(2)^2 + 12(2) + 10]^{3/2}} = \frac{2}{50^{3/2}}.$$

Search the place with maximum curvature

Example Find the point or points on the curve $y = \ln x$ at which the curvature is maximum.

Solution: Note that the domain of the function is $x > 0$. First compute the curvature. As $y' = \frac{1}{x}$ and $y'' = \frac{-1}{x^2}$, we have

$$\kappa(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{1}{x^2} \frac{1}{[1 + \left(\frac{1}{x}\right)^2]^{3/2}} = \frac{|x|}{(x^2 + 1)^{3/2}}.$$

Use calculus to find the maximum of the function $\kappa(x)$. When $x > 0$,

$$\kappa'(x) = \frac{(x^2 + 1)^{3/2} - \frac{3}{2}(x^2 + 1)^{1/2}(2x^2)}{(x^2 + 1)^3} = \frac{\sqrt{x^2 + 1}}{(x^2 + 1)^3}(x^2 + 1 - 3x^2).$$

Therefore, $x = \frac{1}{\sqrt{2}}$ is the only critical point when $x > 0$, and $\kappa'(x) > 0$ if $0 < x < \frac{1}{\sqrt{2}}$ and $\kappa'(x) < 0$ when $x > \frac{1}{\sqrt{2}}$. Therefore, $\kappa(x)$ has maximum value at $x = \frac{1}{\sqrt{2}}$.