## Compute the derivative of vector functions

**Example**: Compute $D_t[\mathbf{u}(t) \cdot \mathbf{v}(t)]$, where $\mathbf{u}(t) = (t, t^2)$ and $\mathbf{v}(t) = (t^2, -t)$.

**Solution (1)**: Apply the product rule for dot products to get

$$\begin{aligned} D_t[(t,t^2) \cdot (t^2, -t)] &= D_t(t,t^2) \cdot (t^2,-t) + (t,t^2) \cdot D_t(t^2,-t) \\ &= (1,2t) \cdot (t^2,-t) + (t,t^2) \cdot (2t,-1) = t^2 - 2t^2 + 2t^2 - t^2 = 0. \end{aligned}$$

**Solution (2)**: First compute the dot product, then compute the derivative:

$$D_t[(t,t^2) \cdot (t^2,-t)] = D_t[t^3 - t^3] = 0.$$

## Find position vectors

**Example** Given the acceleration vector $\mathbf{a}(t) = 6t\mathbf{i} - 5\mathbf{j} + 12t^2\mathbf{k}$, the initial position vector $\mathbf{r}(0) = 3\mathbf{i} + 4\mathbf{j}$, and the initial velocity vector $\mathbf{v}(0) = 4\mathbf{j} - 5\mathbf{k}$, find the position vector $\mathbf{r}(t)$.

**Solution**: First compute the velocity vector

$$\mathbf{v}(t) = \int_0^t \mathbf{a}(t)dt + \mathbf{v}(0) = 3t^2\mathbf{i} - (5t-4)\mathbf{j} + (4t^3 - 5)\mathbf{k}.$$

Then compute $\mathbf{r}(t)$:

$$\mathbf{r}(t) = \int_0^t \mathbf{v}(t) + \mathbf{r}(0) = (t^3+3)\mathbf{i} - \left(\frac{5t^2}{2} - 4t - 4\right)\mathbf{j} + (t^4 - 5t)\mathbf{k}.$$

## Show facts on vectors

**Example** A point moves on a sphere centered at the origin. Show that its velocity vector is always tangent to the sphere.

**Solution**: Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ denote the position vector of the point. Suppose the radius of the sphere is $\sqrt{C}$, then $\mathbf{r}$ satisfies the equation $|\mathbf{r}| = \sqrt{C}$, or $\mathbf{r}^2 = |\mathbf{r}|^2 = C$. Differentiate this equation both sides to get $2\mathbf{r}' \cdot \mathbf{r} = 0$, and so the velocity vector $\mathbf{r}'(t)$ is always perpendicular to the position vector, which implies that the velocity vector $\mathbf{r}'(t)$ is always tangent to the sphere.