Check continuity of functions and determine discontinuities

Facts:

(1) A function $f(x)$ is **continuous** at a point a if a is in the domain of $f(x)$ (that is, $f(a)$ exists as a number) and if $\lim_{x\to a} f(x) = f(a)$. If $f(x)$ is continuous at every point of an interval I, then $f(x)$ is a continuous function over I.

(2) A number a is a discontinuity of a function $f(x)$ if one of the following occurs

(i) a is not in the domain of $f(x)$, or

(ii) if a is in the domain of $f(x)$ but $\lim_{x\to a} f(x)$ does not exist, or

indent (iii) if a is in the domain of $f(x)$ and $\lim_{x\to a} f(x) = L$ exists but $f(a) \neq L$.

(3) Members in the family of continuous functions:

(i) Polynomial, rational functions, power functions, trigonometry functions, logarithm and exponential functions are all continuous in their domains.

(ii) If both $f(x)$ and $g(x)$ are continuous at $x = a$, then each of $f(x)+g(x)$, $(f(x)-g(x)$, $f(x)g(x)$ and (when $g(a) \neq 0$) $f(x)/g(x)$ are all continuous at $x = a$.

(iii) If $g(x)$ is continuous at $x = a$ and $f(x)$ is continuous at $x = g(a)$, then $f \circ g$ is continuous at $x = a$.

(4) A point a is a **removable discontinuity** of a function $f(x)$ if $f(x)$ is not continuous at a but $\lim_{x\to a} f(x) = L$ exists as a finite number. In this case, redefine $f(a) = L$ will make the new function continuous at $x = a$.

Example 1 Apply limit laws to show that the function $h(z) = \sqrt{(z-1)(3-z)}$ is continuous on the interval $[1, 3]$.

Solution: We observe that $[1, 3]$ is the domain of the function $h(z) = \sqrt{(z-1)(3-z)}$. For any point a in $[1, 3]$, by the limit laws (11) and (12) (see "Calculate the limit of a function: Apply Limit Laws and Properties"), we have

$$\lim_{z\to a} \sqrt{(z-1)(3-z)} = \sqrt{\lim_{z\to a}(z-1)(3-z)} = \sqrt{(a-1)(3-a)} = h(a).$$

Thus $\lim_{z\to a} h(z) = h(a)$, and so $h(z)$ is continuous at a. Since a is an arbitrary point in $[1, 3]$, $h(z)$ is continuous in $[1, 3]$.

Example 2 Apply limit laws to show that the function $f(x) = \dfrac{x}{\cos x}$ is continuous on the interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Solution: We observe that $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is in side the domain of the function $f(x) = \dfrac{x}{\cos x}$. For any point a in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, by the limit law (4) (see "Calculate the limit of a function: Apply Limit Laws and Properties"), we have

$$\lim_{x\to a} \frac{x}{\cos x} = \frac{\lim_{x\to a} x}{\lim_{x\to a} \cos x} = \frac{a}{\cos a} = f(a).$$

Thus $\lim_{x\to a} f(x) = h(a)$, and so $f(x)$ is continuous at a. Since a is an arbitrary point in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, $f(x)$ is continuous in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Example 3 Determine where the function $f(x) = x^2 + \dfrac{1}{x}$ is continuous.

Solution: Note that $f(x)$ is the sum of a polynomial x^2 and a rational function $\frac{1}{x}$. Therefore, the domain of $f(x)$ is $(-\infty, 0)$ and $(0, \infty)$. From Facts (3) above, we conclude that $f(x)$ in continuous at every point in $(-\infty, 0)$ and $(0, \infty)$.

Example 4 Determine where the function $f(x) = \dfrac{3}{x^2 - x}$ is continuous.

Solution: Note that $f(x)$ is a rational function $\frac{1}{x}$. Therefore, the domain of $f(x)$ is $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$. From Facts (3) above, we conclude that $f(x)$ in continuous at every point in $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$.

Example 5 Find discontinuities of the function

$$f(x) = \frac{1}{1 - |x|},$$

and tell whether each of them is a removable discontinuity.

Solution: Observe that $x = 1$ and $x = -1$ are the only points not in the domain of $f(x)$, and so these are the only discontinuities of $f(x)$. As both $\lim_{x\to 1} \frac{1}{1-|x|}$ and $\lim_{x\to -1} \frac{1}{1-|x|}$ do not exist (as finite numbers), neither of these two points are removable.

Example 6 Find discontinuities of the function

$$f(x) = \begin{cases} x + 1 & \text{if } x < 1, \\ 3 - x & \text{if } x > 1 \end{cases}$$

and tell whether each of them is a removable discontinuity.

Solution: Observe that $x = 1$ is the only point not in the domain of $f(x)$, and so this is the only discontinuity of $f(x)$. As

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 3 - x = 2 \text{ and } \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x + 1 = 2,$$

we conclude that $\lim x \to 1 f(x) = 2$ and so $x = 1$ is a removable discontinuity of $f(x)$.

Example 7 Find a value of c so that the function

$$f(x) = \begin{cases} 2x + c & \text{if } x \leq 3, \\ 2c - x & \text{if } x > 3 \end{cases}$$

is continuous.

Solution: As polynomials are continuous, we observe that $x = 3$ is the only problematic point. Compute the side limits, we obtain

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 2c - x = 2c - 3 \text{ and } \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 2x + c = 6 + c.$$

In order for $f(x)$ to be continuous at $x = 3$, the limit $\lim x \to 3 f(x)$ must exist, and so both side limits must be the same. Thus we equal the two side limits: $2c - 3 = 6 + c$ to get $c = 9$. Therefore, $c = 9$ is the desired.