

Compute side limits (infinite limits)

When the journey of x in its way $x \rightarrow a$ is restricted to $x > a$ (respectively, $x < a$), then the corresponding limit of $f(x)$ is a **left side limit** (respectively, **right side limit**). The left side limit and right side limit are denoted by

$$\lim_{x \rightarrow a^-} f(x) \text{ and } \lim_{x \rightarrow a^+} f(x)$$

respectively.

(1) The following relationship between limits and side limits is usually applied to check if a limit exists or not:

The limit $\lim_{x \rightarrow a} f(x)$ exists and is equal to a number L if and only if both $\lim_{x \rightarrow a^-} f(x) = L$ and $\lim_{x \rightarrow a^+} f(x) = L$.

(2) The evaluation of side limits is similar to the evaluation of limits. When evaluating side limits of functions involving absolute values, it is recommended that we cover the absolute values into expressions without absolute value signs before evaluating the limit. (See Examples 2, 3 and 4 below).

(3) If in the limiting process of a fraction, as the denominator approaches 0, while the numerator remains or approaches a positive (negative, respectively) nonzero constant, the limiting quantity is positive and can be arbitrarily large (negative with arbitrarily large absolute value, respectively), and so the limit **does not exist**. For convenience, we also write $\lim_{x \rightarrow 1^+} f(x) = \infty$ (or $-\infty$, respectively) to indicate the fact that the limiting quantity can be positively arbitrarily large, (negative with arbitrarily large absolute value, respectively). (**Note that ∞ is not a number**).

Example 1 Compute $\lim_{x \rightarrow 4^-} \sqrt{4-x}$.

Solution: The reason that the restriction $x \rightarrow 4^-$ is needed because the domain of the function $\sqrt{4-x}$ does not allow $x > 4$. The evaluation for this side limit is the same as those for limits.

$$\lim_{x \rightarrow 4^-} \sqrt{4-x} = \lim_{x \rightarrow 4^-} \sqrt{4-4} = \sqrt{0} = 0.$$

Example 2 Compute $\lim_{x \rightarrow 7^-} \frac{7-x}{|x-7|}$.

Solution: When $x \rightarrow 7^-$, x approaches 7 while $x < 7$. As $x < 7$, $|x-7| = 7-x$. Therefore,

we can get rid of the absolute sign before evaluating of the limit.

$$\lim_{x \rightarrow 7^-} \frac{7-x}{|x-7|} = \lim_{x \rightarrow 7^-} \frac{7-x}{7-x} = 1.$$

Example 3 Compute $\lim_{x \rightarrow 7^+} \frac{7-x}{|x-7|}$.

Solution: This problem is given here for your comparison with the previous one. When $x \rightarrow 7^+$, x approaches 7 while $x > 7$. As $x > 7$, $|x-7| = x-7$. Therefore, we can get rid of the absolute sign before evaluating of the limit.

$$\lim_{x \rightarrow 7^-} \frac{7-x}{|x-7|} = \lim_{x \rightarrow 7^-} \frac{7-x}{x-7} = -1.$$

Example 4 Compute $\lim_{x \rightarrow 0^-} \frac{x}{x-|x|}$.

Solution: When $x \rightarrow 0^-$, x approaches 0 while $x < 0$. As $x < 0$, $|x| = -x$. Therefore, we can get rid of the absolute sign before evaluating of the limit.

$$\lim_{x \rightarrow 0^-} \frac{x}{x-|x|} = \lim_{x \rightarrow 0^-} \frac{x}{x-(-x)} = \lim_{x \rightarrow 0^-} \frac{x}{2x} = \lim_{x \rightarrow 0^-} \frac{1}{2} = \frac{1}{2}$$

Example 5 Compute $\lim_{x \rightarrow 1^+} \frac{1-x^2}{|1-x|}$.

Solution: When $x \rightarrow 1^+$, x approaches 1 while $x > 1$. As $x > 1$, $|1-x| = x-1$. Therefore, we can get rid of the absolute sign before evaluating of the limit.

$$\lim_{x \rightarrow 1^+} \frac{1-x^2}{|1-x|} = \lim_{x \rightarrow 1^+} \frac{(1-x)(1+x)}{x-1} = \lim_{x \rightarrow 1^+} \frac{1+x}{-1} = -2.$$

Example 6 Compute $\lim_{x \rightarrow 1^+} \frac{\sqrt{x-1}}{|1-x|}$.

Solution: When $x \rightarrow 1^+$, x approaches 1 while $x > 1$. As $x > 1$, $|1-x| = x-1$. Therefore, we can get rid of the absolute sign before evaluating of the limit. We end up with the limit $\lim_{x \rightarrow 1^+} \frac{1}{\sqrt{x-1}}$ in the last step. As the denominator approaches 0, while the numerator remains as a positive nonzero constant, the limiting quantity can be arbitrarily large and so the limit **does not exist**. For convenience, we also write $\lim_{x \rightarrow 1^+} \frac{1}{\sqrt{x-1}} = \infty$ to indicate the fact that the limiting quantity can be arbitrarily large. (**Note that ∞ is not a number**).

$$\lim_{x \rightarrow 1^+} \frac{\sqrt{x-1}}{|1-x|} = \lim_{x \rightarrow 1^+} \frac{\sqrt{x-1}}{x-1} = \lim_{x \rightarrow 1^+} \frac{1}{\sqrt{x-1}} = \infty.$$

Example 7 The function $f(x) = \frac{1-x^2}{x+2}$ has exactly one point a where both side limits fail to exist. Describe the behavior of $f(x)$ for x near a .

Solution: We observe that both the numerator and the denominator are polynomials, and that only $x = -2$ is not in the domain of $f(x)$. We therefore consider the side limits at $a = -2$.

When considering $\lim_{x \rightarrow -2^+} \frac{1-x^2}{x+2}$, we observe that as $x \rightarrow -2^+$, $x > -2$ and so the denominator is positive and approaches 0, while the numerator approaches $1 - (-2)^2 < 0$. Therefore, in this process, the fraction is negative in sign and the absolute value of it can be arbitrarily large. Hence we have

$$\lim_{x \rightarrow -2^+} \frac{1-x^2}{x+2} = -\infty.$$

When considering $\lim_{x \rightarrow -2^-} \frac{1-x^2}{x+2}$, we observe that as $x \rightarrow -2^-$, $x < -2$ and so the denominator is negative and approaches 0, while the numerator approaches $1 - (-2)^2 < 0$. Therefore, in this process, the fraction is positive in sign and the absolute value of it can be arbitrarily large. Hence we have

$$\lim_{x \rightarrow -2^-} \frac{1-x^2}{x+2} = \infty.$$

Summing up, the point a is -2 . When x approaches -2 from the left side, $f(x)$ tends to ∞ ; when x approaches -2 from the right side, $f(x)$ tends to $-\infty$.