Compute limits: using the Squeeze Law

The Squeeze Law Suppose that $f(x) \le g(x) \le h(x)$ for all $x \ne a$ in some neighborhood of a and laws that

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x).$$

Then

$$\lim_{x\to a} g(x) = L.$$

Example 1 Compute $\lim_{x\to 0} x^2 \sin \frac{1}{x^2}$.

Solution: To apply the Squeeze Law, we view $a = 0$ and

$$g(x) = x^2 \sin \frac{1}{x^2}.$$

The key idea is to find the appropriate $f(x)$ and $h(x)$. Note that $-1 \le \sin\theta \le 1$, for any θ (in this occasion $\theta = \frac{1}{x^2}$). Therefore, we always have, when $x \ne 0$,

$$-x^2 \le x^2 \sin \frac{1}{x^2} \le x^2.$$

Choose $f(x) = -x^2$ and $h(x) = x^2$, and note that both $\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2$. Therefore, by Squeeze Law,

$$\lim_{x\to 0} x^2 \sin \frac{1}{x^2} = 0.$$

Example 2 Compute $\lim_{x\to 0} \sqrt[3]{x} \sin \frac{1}{x}$.

Solution: To apply the Squeeze Law, we view $a = 0$ and

$$g(x) = \sqrt[3]{x} \sin \frac{1}{x}.$$

The key idea is to find the appropriate $f(x)$ and $h(x)$. Note that $-1 \le \sin\theta \le 1$, for any θ (in this occasion $\theta = \frac{1}{x}$). Therefore, we always have, when $x \ne 0$, (absolute values must be added so that when this positive amount multiplied to both sides of an inequality, the inequality sign will not be reversed).

$$-|\sqrt[3]{x}| \le \sqrt[3]{x} \sin \frac{1}{x} \le |\sqrt[3]{x}|.$$

Choose $f(x) = -|\sqrt[3]{x}|$ and $h(x) = |\sqrt[3]{x}|$, and note that both $\lim_{x\to 0} -|\sqrt[3]{x}| = 0 = \lim_{x\to 0} |\sqrt[3]{x}|$. Therefore, by Squeeze Law,

$$\lim_{x\to 0} \sqrt[3]{x} \sin \frac{1}{x} = 0.$$