Compute a limit by using the basic trigonometric limit

The basic trigonometric limit

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1.$$

Using the basic limit, we can also have another form of the basic limit.

$$\lim_{\theta\to 0}\frac{1}{\frac{\sin\theta}{\theta}}=\frac{1}{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}=\frac{1}{1}=1.$$

Example 1 Compute $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

Solution: The key idea is to convert the limit into one that the basic limit can be applied. From trigonometry, we recall that $\sin x$ and $\cos x$ are related by an identity $\sin^2 x+\cos^2 x=1$. Therefore, use limit laws and $\lim_{x\to 0}\cos x=1$ to get

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x}{x^2} &= \lim_{x\to 0}\frac{1-\cos x}{x^2}\frac{1+\cos x}{1+\cos x}\\
&= \lim_{\theta\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2}\frac{1}{1+\cos x}=\lim_{\theta\to 0}\frac{1-\cos^2 x}{x^2}\frac{1}{1+\cos x}\\
&= \lim_{\theta\to 0}\frac{\sin^2 x}{x^2}\frac{1}{1+\cos x}=\lim_{\theta\to 0}\frac{\sin x}{x}\lim_{\theta\to 0}\frac{\sin x}{x}\lim_{\theta\to 0}\frac{1}{1+\cos x}\\
&= 1\cdot 1\cdot\frac{1}{1+1}=\frac{1}{2}.
\end{aligned}$$

Example 2 Compute $\lim_{x\to 0}\frac{\sin(ax)}{ax}$, where $a\neq 0$ is a real constant.

Solution: The key idea is to convert the limit into one that the basic limit can be applied. Let $u=ax$. Then when $x\to 0$, $u=ax\to 0$. Therefore,

$$\lim_{x\to 0}\frac{\sin(ax)}{ax}=\lim_{u\to 0}\frac{\sin(u)}{u}=1.$$

Example 3 Compute $\lim_{x\to 0}\frac{\sin(2x^2)}{x^2}$.

Solution: The key idea is to convert the limit into one that the basic limit can be applied. Let $u=2x^2$. Then $x^2=\frac{u}{2}$, and when $x\to 0$, $u=2x^2\to 0$. Therefore,

$$\lim_{x\to 0}\frac{\sin(2x^2)}{x^2}=\lim_{u\to 0}\frac{2\sin(u)}{u}=\lim_{u\to 0}2\frac{\sin(u)}{u}=2.$$

Example 4 Compute $\lim_{x\to 0} \dfrac{\tan(3x)}{\tan(5x)}$.

Solution: The key idea is to convert the limit into one that the basic limit can be applied. To do that, use $\tan\theta = \frac{\sin\theta}{\cos\theta}$ (here $\theta = 3x$ and $5x$, respectively) to get the sine functions out from the tangent functions. Thus, apply Example 2 above with $a = 3$ and $a = 5$, respectively, and use $\lim_{x\to 0} \cos ax = 1$ to get

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan(3x)}{\tan(5x)} &= \lim_{x\to 0} \frac{\sin(3x)}{\cos(3x)} \frac{\cos(5x)}{\sin(5x)} \\
&= \lim_{x\to 0} \frac{\sin(3x)}{3x} \frac{3x}{\cos(3x)} \frac{\cos(5x)}{5x} \frac{5x}{\sin(5x)} \\
&= \lim_{x\to 0} \frac{\sin(3x)}{3x} \lim_{x\to 0} \left[\frac{3}{\cos(3x)} \frac{\cos(5x)}{5}\right] \lim_{x\to 0} \frac{5x}{\sin(5x)} \\
&= 1 \cdot \frac{3}{5} \cdot 1 = \frac{3}{5}.
\end{aligned}$$

Example 5 Compute $\lim_{x\to 0} x \csc x \sec x$.

Solution: The key idea is to convert the limit into one that the basic limit can be applied. To do that, use $\csc x \frac{1}{\sin x}$ to get the sine function out from the co secant function. Thus, use $\lim_{x\to 0} \sec x = 1$ to get

$$\lim_{x\to 0} x \csc x \sec x = \lim_{x\to 0} \frac{x}{\sin x} \sec x = \lim_{x\to 0} \frac{x}{\sin x} \lim_{x\to 0} \sec x = 1.$$