Calculate the limit of a function: cancelling zero factors before evaluating for the $\frac{0}{0}$ type limits, and other hard to decide situations

When facing a $\frac{0}{0}$ type limit, one must try to get rid of the zero factor(s) **before** evaluating the limit. In any case, $\frac{0}{0}$ is **always a wrong answer**.

Example 1 Find $\lim_{x\to 1} \frac{x-1}{x^2-1}$.

Solution: When $x \to 1$, both $x-1$ and x^2-1 go to 0, and so this is a $\frac{0}{0}$ type limit. Factor $x^2-1=(x-1)(x+1)$. Then cancel the zero factors before evaluating the limit:

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{x-1}{(x-1)(x+1)} = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}.$$

Example 2 Find $\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2}$.

Solution: When $x \to 0$, both $\sqrt{x^2+9}-3$ and x^2 go to 0, and so this is a $\frac{0}{0}$ type limit.

In order to cancel the zero factors, we need to "free" x^2+9 out of the radical. Apply the algebraic identity $(A+B)(A-B)=A^2-B^2$ with $A=\sqrt{x^2+9}$ and $B=3$ to get

$$\sqrt{x^2+9}-3 = \frac{(\sqrt{x^2+9}-3)(\sqrt{x^2+9}+3)}{\sqrt{x^2+9}+3} = \frac{x^2+9-9}{\sqrt{x^2+9}-3} = \frac{x^2}{\sqrt{x^2+9}-3}.$$

Therefore,

$$\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2} = \lim_{x\to 0} \frac{x^2}{x^2(\sqrt{x^2+9}+3)} = \lim_{x\to 0} \frac{1}{\sqrt{x^2+9}+3} = \frac{1}{6}.$$

Example 3 Find $\lim_{x\to 1} \left[\frac{1}{x-1} - \frac{2}{x^2-1}\right]$.

Solution: When $x \to 1$, both $x-1$ and x^2-1 go to 0, and so this is a $\infty - \infty$ type limit. One can use algebra to convert it into a $\frac{0}{0}$ type limit, by combining the two fractions, as follows.

$$\frac{1}{x-1} - \frac{2}{x^2-1} = \frac{(x+1)-2)}{x^2-1} = \frac{x-1}{x^2-1}.$$

This becomes the problem of Example 1.