Arc length and surface area computing

1. Arc length computing. Using the a line segment to approximate a small arc piece, the length of the small arc piece can be approximated by

$$ds \simeq \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

The total length of the whole arc can then be obtained by adding up all lengths of the small arc pieces in the Riemann sum sense under a limiting process, which leads to (with x bounds given as an example)

$$\text{Arc length } = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

2. Surface area computing. The surface generated by rotating a smooth arc C around an axis. Then area of the corresponding surface generated by a small piece of arc with length ds equals

$$dA = 2\pi \text{ (distance from the arc piece to the ration axis)}ds.$$

The area of the whole surface is then (assuming the axis of rotation is parallel to the x-axis, and the arc C has x bounds a and b)

$$\text{Area } = 2\pi \int_a^b \text{ (distance from the arc piece to the ration axis)}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

If the axis of rotation is parallel to the y-axis, then changes should be made accordingly.

Example 1 Find the length of an arc C which is given by $y = \frac{1}{6}x^3 + \frac{1}{2x}$ from $x = 1$ to $x = 3$.

Solution: First compute $\frac{dy}{dx}$, and ds:

$$\frac{dy}{dx} = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}.$$

Thus

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{2}x^2 - \frac{1}{2}x^{-2}\right)^2 = \frac{1}{4}x^4 - \frac{2}{4} + \frac{1}{4}x^{-4},$$

and so

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{\frac{(x^2 + x^{-2})^2}{4}} dx = \frac{x^2 + x^{-2}}{2} dx$$

It follows that

$$\text{Arc length } = \int_1^3 \frac{x^2 + x^{-2}}{2} dx = \frac{14}{3}.$$

Example 2 Find the length of an arc C which is given by $x = \frac{2}{3}(y-1)^{\frac{3}{2}}$ from $y = 1$ to $y = 5$.

Solution: First compute $\frac{dx}{dy}$, and ds:

$$\frac{dx}{dy} = \frac{2}{3}\frac{3}{2}(y-1)^{\frac{1}{2}}, \text{ and so } \left(\frac{dx}{dy}\right)^2 = y - 1.$$

Thus

$$\text{Arc length } = \int_1^5 \sqrt{1 + (y-1)}dy = \left[\frac{2y^{\frac{3}{2}}}{3}\right]_1^5 = \frac{10\sqrt{5} - 2}{3}.$$

Example 3 Find the area of the surface of revolution generated by revolving the curve C which is given by $y = x^3$, $1 \le x \le 2$ about the x-axis.

Solution: First compute $\frac{dy}{dx}$, and ds:

$$\frac{dy}{dx} = 3x^2, \text{ and so } ds = \sqrt{1 + 9x^4}dx.$$

For each x with $1 \le x \le 2$, the distance from the corresponding arc piece to the axis of rotation is x^3. Thus

$$\text{Surface area } = 2\pi \int_1^2 x^3 \sqrt{1 + 9x^4}dx = \frac{\pi}{27}(145\sqrt{145} - 10\sqrt{10}).$$

Example 4 Set up and simplify the integral that gives the surface area of revolution generated by rotation of the curve $y = x^2$, $0 \le x \le 4$ about the y-axis. (No need to evaluate the integral.)

Solution: First compute $\frac{dy}{dx}$, and ds:

$$\frac{dy}{dx} = 2x, \text{ and so } ds = \sqrt{1 + 4x^2}dx.$$

For each x with $0 \le x \le 4$, the distance from the corresponding arc piece to the axis of rotation is x. Thus

$$\text{Surface area } = 2\pi \int_0^4 x\sqrt{1 + 4x^2}dx.$$

Example 5 Set up and simplify the integral that gives the surface area of revolution generated by rotation of the curve $y = x^2$, $0 \le x \le 4$ about the x-axis. (No need to evaluate the integral.)

Solution: First compute $\frac{dy}{dx}$, and ds:

$$\frac{dy}{dx} = 2x, \text{ and so } ds = \sqrt{1 + 4x^2}dx.$$

For each x with $0 \le x \le 4$, the distance from the corresponding arc piece to the axis of rotation is y which is x^2. Thus

$$\text{Surface area } = 2\pi \int_0^4 x^2\sqrt{1 + 4x^2}dx.$$

Example 6 Set up and simplify the integral that gives the surface area of revolution generated by rotation of the curve $y = x^2$, $0 \leq x \leq 4$ about the line $x = 2$. (No need to evaluate the integral.)

Solution: First compute $\frac{dy}{dx}$, and ds:

$$\frac{dy}{dx} = 2x, \text{ and so } ds = \sqrt{1 + 4x^2}dx.$$

For each x with $0 \leq x \leq 4$, the distance from the corresponding arc piece to the axis of rotation is $2 - x$. Thus

$$\text{Surface area } = 2\pi \int_0^4 (2 - x)\sqrt{1 + 4x^2}dx.$$

Example 7 Set up and simplify the integral that gives the surface area of revolution generated by rotation of the curve $y = x^2$, $0 \leq x \leq 4$ about the line $y = 4$. (No need to evaluate the integral.)

Solution: First compute $\frac{dy}{dx}$, and ds:

$$\frac{dy}{dx} = 2x, \text{ and so } ds = \sqrt{1 + 4x^2}dx.$$

For each x with $0 \leq x \leq 4$, the distance from the corresponding arc piece to the axis of rotation is $4 - x^2$. Thus

$$\text{Surface area } = 2\pi \int_0^4 (4 - x^2)\sqrt{1 + 4x^2}dx.$$