Volume Computing

1. Cross section technique. A solid T is placed along an axis (x-axis or y-axis, assume it to be x-axis) from lower bound a and upper bound b. If for each value x between a and b, the area of the cross section of T is $A(x)$, then

$$\text{Volume of } T = \int_a^b A(x)dx.$$

Note that $A(x)dx$ represents the volume of a small slice of T while $\int_a^b A(x)dx$ means adding up all such small pieces in the Riemann sum sense under a limiting process.
Determine the bounds of integration. These bounds are the coordinates of the ends of the solid. Suppose a solid T is obtained from rotating a region R about an axis. If the axis of rotation is parallel to the x-axis, then the integration bounds are x-bounds; If the axis of rotation is parallel to the y-axis, then the integration bounds are y-bounds.

2. Cylindrical shell technique. A cylindrical shell with radius r, height h and thickness Δ has volume $2\pi dh\Delta$. To apply the shell technique to compute a volume, we first partition the solid into small shells and then add up all the volumes of the shells in the Riemann sum sense under a limiting process. Therefore, a generic form of the shell technique is

$$\int_a^b 2\pi \text{ (distance from the shell to the axis of rotation) (height of the shell) (thickness).}$$

Determine the bounds of integration. This is different from the cross section technique. Suppose a solid T is obtained from rotating a region R about an axis. If the axis of rotation is parallel to the x-axis, then the integration bounds are y-bounds; If the axis of rotation is parallel to the y-axis, then the integration bounds are x-bounds.

Example 1 Find volume of the solid obtained by rotating the region R bounded by $y = 9 - x^2$ and $y = 0$ about x-axis.

Solution: We use cross section technique. First determine the integration bounds. As the axis of rotation is the x-axis, the bounds should be the x-coordinated of the ends of R. Note that the curves $y = 9 - x^2$ and $y = 0$ intersect at $x = -3$ and $x = 3$, and so lower bound $a = -3$ and upper bound $b = 3$.

For each x with $-3 \le x \le 3$, the cross section is a circle with radius $9 - x^2$, (the y value of the curve bounded above region R at x), and so A$A(x) = \pi(9 - x^2)^2$. Thus

$$\text{Volume } = \pi \int_{-3}^{3} (9 - x^2)^2 dx = \pi \int_{-3}^{3} (81 - 18x^2 + x^4)dx = \frac{1296}{5}\pi.$$

Example 2 Find volume of the solid obtained by rotating the region R bounded by $y = 1 - x^2$ and $y = 0$ about the line $x = 2$.

Cross Section Solution: As the axis of rotation is parallel to the y-axis, the bounds should be the y-coordinated of the ends of R. Note that the curves $y = 1 - x^2$ and $y = 0$ bound the region from above and from below, respectively, and so lower bound $c = 0$ and upper bound $d = 1$.

For each y with $0 \leq y \leq 1$, the cross section of the solid at y is an annular ring with the bigger radius $r_2 = 2 + \sqrt{1-y}$ and smaller radius $r_1 = 2 - \sqrt{1-y}$. Thus the cross section area at y is

$$A(y) = \pi r_2^2 - \pi r_1^2 = \pi(r_2^2 - r_1^2) = \pi\left[(2+\sqrt{1-y})^2 - (2-\sqrt{1-y})^2\right] = 8\pi\sqrt{1-y}.$$

It follows that the volume of the solid is

$$\text{Volume } = 8\pi \int_0^1 \sqrt{1-y}dy = 8\pi\left[\frac{2(1-y)^{\frac{3}{2}}}{3}\right]_0^1 = \frac{16}{3}\pi.$$

Shell Technique Solution: As the axis of rotation is parallel to the y-axis, the bounds should be the x-coordinated of the ends of R for the shell technique. Note that the curves $y = 1 - x^2$ and $y = 0$ intersect at $x = -1$ and $x = 1$, and so lower bound $a = -1$ and upper bound $b = 1$.

For each x with $-1 \leq x \leq 1$, the shell generated at x has radius $2 - x$, height $1 - x^2$, and thickness dx, and so the volume of this shell at x is

$$2\pi(2-x)(1-x^2)dx = 2\pi(2 - x - 2x^2 + x^3)dx.$$

It follows that the volume of the solid is (using properties of even and odd functions integrating on a symmetric interval)

$$\text{Volume } = 2\pi \int_{-1}^1 (2 - x - 2x^2 + x^3)dx = 4\pi\left[2x - \frac{2x^3}{3}\right]_0^1 = \frac{16}{3}\pi.$$