Fundamental Theorem of Calculus

Fundamental Theorem of Calculus (FTC) Let $f(x)$ be a continuous function on the interval $[a, b]$.
Part 1: If $F(x)$ on $[a, b]$ is defined by

$$F(x) = \int_a^x f(t)dt,$$

then $F'(x) = f(x)$, for x in $[a, b]$.
Part 2: If $G(x)$ on $[a, b]$ satisfies $G'(x) = f(x)$, then

$$\int_a^b f(t)dt = G(b) - G(a).$$

Example 1 Find the the derivative of

$$F(x) = \int_{-1}^x (t^2 + 1)^{17} dt.$$

Solution: Here $f(t) = (t^2 + 1)^{17}$ is a continuous function on $(-\infty, \infty)$. By the FTC,

$$F'(x) = f(x) = (x^2 + 1)^{17}.$$

Example 2 Find the the derivative of

$$F(x) = \int_x^{10} \sqrt{t^2 + 1} dt.$$

Solution: Note that the variable bound is the lower bound in the definition of $F(x)$. Before applying FCT, we utilize integral properties to change the bounds into the form for which FCT is applicable.

$$F(x) = \int_x^{10} \sqrt{t^2 + 1} dt = -\int_{10}^x \sqrt{t^2 + 1} dt = \int_{10}^x -\sqrt{t^2 + 1} dt.$$

Now $f(t) = -\sqrt{t^2 + 1}$ is a continuous function on $(-\infty, \infty)$. Apply FCT to get

$$F'(x) = f(x) = -\sqrt{x^2 + 1}.$$

Example 3 Find the the derivative of

$$F(x) = \int_0^{\sin x} \sqrt{t^2 + 1} dt.$$

Solution: Note that the variable bound is is a function instead of simply an "x". Therefore, we cannot directly apply FCT. Set $u = \sin x$. The we have a function

$$G(u) = \int_0^u \sqrt{t^2+1}dt,$$

for which FCT is applicable with $G'(u) = \sqrt{u^2+1}$.

But what we want is $F'(x)$. As $F(x) = G(u(x))$, we can use Chain Rule $F'(x) = G'(u)u'(x)$ to get the job done:

$$F(x) = G'(u)u'(x) = \sqrt{u^2+1}\cos x = \sqrt{\sin^2 x + 1}\cos x.$$

Example 4 Find the the derivative of

$$F(x) = \int_0^{x^2} \sin(t^2)dt.$$

Solution: Note that the variable bound is is a function instead of simply an "x". Therefore, we cannot directly apply FCT. Set $u = x^2$. The we have a function

$$G(u) = \int_0^u \sin(t^2)dt,$$

for which FCT is applicable with $G'(u) = \sin(u^2)$.

But what we want is $F'(x)$. As $F(x) = G(u(x))$, we can use Chain Rule $F'(x) = G'(u)u'(x)$ to get the job done:

$$F(x) = G'(u)u'(x) = \sin(u^2)(2x) = 2x\sin(x^4).$$