Average of a function

(1) If $f(x)$ is integrable on $[a, b]$, then the **average value** $\bar{y}$ of $y = f(x)$ for x in the interval $[a, b]$ is

$$\bar{y} = \frac{1}{b - a} \int_a^b f(x)dx.$$

(2) Average Value Theorem: If $f(x)$ is integrable on $[a, b]$, then there exists a point $\bar{x}$ in the interval $[a, b]$ such that

$$f(\bar{x}) = \frac{1}{b - a} \int_a^b f(x)dx.$$

Example 1 Find the average value $\bar{y}$ of the function $f(x) = x^2$ on the interval $[-4, 4]$; and find a point $\bar{x}$ in the interval $[-4, 4]$ such that $f(\bar{x}) = \bar{y}$.

Solution: First we compute $\bar{y}$:

$$\bar{y} = \frac{1}{4 - (-4)} \int_{-4}^4 x^2 dx = \frac{1}{8}\left[\frac{x^3}{3}\right]_{-4}^4 = \frac{1}{8}\left[\frac{64}{3} - \frac{-64}{3}\right] = \frac{16}{3}.$$

To find $\bar{x}$, solve the equation $f(x) = \bar{y}$ for x. That is, solve

$$x^2 = \frac{16}{3},$$

for x. It follows that $x = \pm\frac{4}{\sqrt{3}}$. Either value can be $\bar{x}$, as both are in the interval $[-4, 4]$.

Example 2 Find the average value $\bar{y}$ of the function $f(x) = \sin(2x)$ on the interval $[0, \frac{\pi}{2}]$.

Solution: First we compute $\bar{y}$:

$$\bar{y} = \frac{1}{\frac{\pi}{2} - 0} \int_0^{\frac{\pi}{2}} \sin(2x)dx = -\frac{2}{\pi}\left[\frac{\cos(2x)}{2}\right]_0^{\frac{\pi}{2}} = -\frac{2}{\pi}\left[\frac{\cos(\pi)}{2} - \frac{\cos(0)}{2}\right] = \frac{2}{\pi}.$$

Example 3 Find the average value $\bar{y}$ of the function $f(x) = |x|$ on the interval $[-1, 2]$.

Solution: To compute $\bar{y}$, we need to evaluate the integral $\int_{-1}^2 |x|dx$. Draw the graph of $y = |x|$ on $[-1, 2]$ we find that the graph of $y = |x|$, the vertical lines $x = -1$, $x = 2$, and the x-axis form two right triangles with area equaling $\frac{1}{2}$ and 2, respectively. Thus :

$$\bar{y} = \frac{1}{2 - (-1)} \int_{-1}^2 |x|dx = \frac{1}{3}\left(\frac{1}{2} + 2\right) = \frac{1}{3}\left(\frac{5}{2}\right) = \frac{5}{6}.$$