Evaluation of integrals

(1) Evaluation Formula: If $f(x)$ is continuous and if $G'(x) = f(x)$, then

$$\int_a^b f(x)dx = G(b) - G(a).$$

(2). Properties of Integrals:
(i) $\int_a^b c dx = c(b-a)$ and $\int_b^a f(x)dx = -\int_a^b f(x)dx$.
(ii) (**Linear Properties**) For a constant c, $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ and $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.
(iii) (**Interval Union Property**) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$. (See Examples 4 and 5 for applications of this property).
(iv) If $f(x) \le g(x)$ for all x in $[a, b]$, then

$$\int_a^b f(x)dx \le \int_a^b g(x)dx.$$

(v) If $m \le f(x) \le M$ for all x in $[a, b]$, then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a).$$

Example 1 Evaluate $\displaystyle\int_1^3 \frac{x^4+1}{x^2}dx$.

Solution:

$$\begin{aligned}\int_1^3 \frac{x^4+1}{x^2}dx &= \int_1^3 \left(\frac{x^4}{x^2} + \frac{1}{x^2}\right)dx = \int_1^3 (x^2 + x^{-2})dx = \left[\frac{x^3}{3} + \frac{x^{-1}}{-1}\right]_1^3 \\ &= \left(\frac{27}{3} - \frac{1}{3}\right) - \left(\frac{1}{3} - 1\right) = 10 - \frac{2}{3} = \frac{28}{3}.\end{aligned}$$

Example 2 Evaluate $\displaystyle\int_1^3 (x-1)^5 dx$.

Solution 1: First set $u = x - 1$ $(du = dx)$ to compute an antiderivative

$$\int (x-1)^5 dx = \int u^5 du = \frac{u^6}{6} = \frac{(x-1)^6}{6}.$$

Then apply the Evaluation Formula:

$$\int_1^3 (x-1)^5 dx = \left.\frac{(x-1)^6}{6}\right|_1^3 = \frac{2^6}{6} - 0 = \frac{32}{3}.$$

Solution 2: Set $u = x - 1$. Then $du = dx$ and the integration bounds from 1 to 3 for x become 0 to 2 for u accordingly. Therefore,

$$\int_1^3 (x-1)^5 dx = \int_0^2 u^5 du = \left.\frac{u^6}{6}\right|_0^2 = \frac{2^6}{6} - 0 = \frac{32}{3}.$$

Example 3 Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sin x \cos x dx$.

Solution 1: First set $u = \sin x$ ($du = \cos x dx$) to compute an antiderivative

$$\int \sin x \cos x dx = \int u du = \frac{u^2}{2} = \frac{\sin^2 x}{2}.$$

Then apply the Evaluation Formula (note that $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\sin 0 = 0$):

$$\int_0^{\frac{\pi}{4}} \sin x \cos x dx = \left.\frac{\sin^2 x}{2}\right|_0^{\frac{\pi}{4}} = \frac{2}{4} - 0 = \frac{1}{2}.$$

Solution 2: Set $u = \sin x$. Then $du = \cos x dx$ and the integration bounds from 0 to $\frac{\pi}{4}$ for x become 0 to $\frac{\sqrt{2}}{2}$ for u accordingly. Therefore,

$$\int_0^{\frac{\pi}{4}} \sin x \cos x dx = \int_0^{\frac{\sqrt{2}}{2}} u du = \left.\frac{u^2}{2}\right|_0^{\frac{\sqrt{2}}{2}} = \frac{2}{4} - 0 = \frac{1}{2}.$$

Example 4 Evaluate $\displaystyle\int_0^2 |x - \sqrt{x}| dx$.

Solution: The difficulty of this exercise is the handling of the absolute value. We must first get rid of the absolute value sign and then apply the Interval Union Property to evaluate the integral, as shown below.

Step 1: Get rid of the absolute sign by determining when the related quantity is non negative. When $x - \sqrt{x} \geq 0$? We consider the inequality $x \geq \sqrt{x}$ and square both sides (as $x \geq 0$, this can be done) to get $x^2 \geq x$, or equivalently, $x^2 - x \geq 0$. Factor it to get

$$x(x-1) \geq 0.$$

Since $x \geq 0$, we must have $x - 1 \geq 0$. Thus we conclude that when $0 \leq x \leq 2$, $x - \sqrt{x} \geq 0$ if and only if $x \geq 1$. Therefore,

$$|x - \sqrt{x}| = \begin{cases} -(x - \sqrt{x}) & \text{if } 0 \leq x \leq 1 \\ x - \sqrt{x} & \text{if } 1 \leq x \leq 2 \end{cases}.$$

Step 2: Use Interval Union Property to evaluate the integral. Use the findings in Step 1, we can get rid of the absolute signs in the evaluation of the integral.

$$\begin{aligned}
\int_0^2 |x-\sqrt{x}|dx &= \int_0^1 |x-\sqrt{x}|dx + \int_1^2 |x-\sqrt{x}|dx \\
&= -\int_0^1 (x-\sqrt{x})dx + \int_1^2 (x-\sqrt{x})dx \\
&= -\left[\frac{x^2}{2} - \frac{2x^{\frac{3}{2}}}{3}\right]_0^1 + \left[\frac{x^2}{2} - \frac{2x^{\frac{3}{2}}}{3}\right]_1^2 \\
&= -\left[\frac{1}{2} - \frac{2}{3}\right] - 0 + \left[\frac{4}{2} - \frac{4\sqrt{2}}{3}\right] - \left[\frac{1}{2} - \frac{2}{3}\right] \\
&= \frac{1}{6} + \frac{6-4\sqrt{2}}{3} + \frac{1}{6} = \frac{7-4\sqrt{2}}{3}.
\end{aligned}$$

Example 5 Evaluate $\int_{-2}^{2} |x^2-1|dx$.

Solution: The difficulty of this exercise is the handling of the absolute value. We must first get rid of the absolute value sign and then apply the Interval Union Property to evaluate the integral, as shown below.

Step 1: Get rid of the absolute sign by determining when the related quantity is non negative. When $x^2-1 \geq 0$? Consider the graph $y = x^2-1$ to see that when $-2 \leq x \leq 2$, $x^2-1 \geq 0$ if and only if $-2 \leq x \leq -1$ and $1 \leq x \leq 2$. Therefore,

$$|x^2-1| = \begin{cases} x^2-1 & \text{if } -2 \leq x \leq -1 \\ -(x^2-1) & \text{if } -1 \leq x \leq 1 \\ x^2-1 & \text{if } 1 \leq x \leq 2 \end{cases}.$$

Step 2: Use Interval Union Property to evaluate the integral. Use the findings in Step 1, we can get rid of the absolute signs in the evaluation of the integral.

$$\begin{aligned}
& \int_{-2}^{2} |x^2-1|dx \\
= & \int_{-2}^{-1} (x^2-1)dx + \int_{-1}^{1} -(x^2-1)dx + \int_{1}^{2} (x^2-1)dx \\
= & \int_{-2}^{-1} (x^2-1)dx - \int_{-1}^{1} (x^2-1)dx + \int_{1}^{2} (x^2-1)dx \\
= & \left[\frac{x^3}{3} - x\right]_{-2}^{-1} - \left[\frac{x^3}{3} - x\right]_{-1}^{1} + \left[\frac{x^3}{3} - x\right]_{1}^{2} \\
= & \left(-\frac{1}{3} - (-1)\right) - \left(\frac{-8}{3} - (-2)\right) - \left(\frac{1}{3} - 1\right) + \left(\frac{-1}{3} - (-1)\right) + \left(\frac{8}{3} - 1\right) - \left(\frac{1}{3} - 1\right) \\
= & 5.
\end{aligned}$$