Calculate the limit of a function: Apply Limit Laws and Properties

Limit Laws and Properties: Suppose that c is a constant, and that the limits

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x)$$

exist. Then

(1) $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.

(2) $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.

(3) $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$.

(4) $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.

(5) When $\lim_{x\to a} g(x) \neq 0$,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

(6) When n is a positive number,

$$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n.$$

(7) $\lim_{x\to a} c = c$.

(8) $\lim_{x\to a} x = a$.

(9) When $n > 0$ is an integer, $\lim_{x\to a} x^n = a^n$.

(10) When $n > 0$ is an integer,

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}, \text{ when } n \text{ is even, we assume } a \geq 0.$$

(11) When $n > 0$ is an integer,

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}, \text{ when } n \text{ is even, we assume } \lim_{x\to a} f(x) \geq 0.$$

(12) If f is a polynomial or a rational function, and if a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a).$$

Example 1 Find $\lim_{x\to 8}(2x^2 - 3x + 4)$ and justify each step.

Solution: There are two ways of doing so. The first one applies (2) and (4) in Step 1, (3) in Step 2, and (9), (8), (7) in Step 3, as shown below.

$$\lim_{x\to 8}(2x^2 - 3x + 4) = \lim_{x\to 8}(2x^2) - \lim_{x\to 8}(3x) + \lim_{x\to 8} 4$$

$$
\begin{aligned}
&= 2\lim_{x\to 8} x^2 - 3\lim_{x\to 8} x + \lim_{x\to 8} 4 \\
&= 2(5^2) - 3(5) + 4 \\
&= 39.
\end{aligned}
$$

Applying Limit Laws 1-5 on polynomials or fractional functions actually yields Property (12). Therefore, one can apply (12) and get more directly

$$\lim_{x\to 8}(2x^2 - 3x + 4) = 2(5^2) - 3(5) + 4 = 39.$$

Example 2 Find $\lim_{x\to 1}[\sqrt[5]{x^2 - x} + (x^3 + x)^9]$ and justify each step.

Solution: Apply (1) in Step 1, (11) and (6) in Step 2, (2) and (1) in Step 3, and (9) and (8) in Step 4, as shown below.

$$
\begin{aligned}
\lim_{x\to 1}[\sqrt[5]{x^2 - x} + (x^3 + x)^9] &= \lim_{x\to 1} \sqrt[5]{x^2 - x} + \lim_{x\to 1}(x^3 + x)^9 \\
&= \sqrt[5]{\lim_{x\to 1} x^2 - x} + [\lim_{x\to 1} x^3 + \lim_{x\to 1} x]^9 \\
&= \sqrt[5]{1^2 - 1} + [1^3 + 1]^9 \\
&= 2^9 = 512.
\end{aligned}
$$