Riemann Sums and definite integrals

(1). Riemann Sums For a function f defined on $[a, b]$, a **partition** P of $[a, b]$ into a collection of subintervals

$$[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n],$$

and for each $i = 1, 2, \cdots, n$, a point x_i^* in $[x_{i-1}, x_i]$, the sum

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

is called a **Riemann sum** for f determine by the partition P. Let $|P| = \max\{x_i - x_{i-1}$ for all $i = 1, 2, \cdots, n\}$ denote the longest length of all the subintervals.

(2). The Definite Integral The **definite integral** of f from a to b is the number

$$\int_a^b f(x)dx = \lim_{|P| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

provided the limit exists. (We in this case say f is **integrable** on $[a, b]$).

(3). Computing Riemann Sums For a continuous function f on $[a, b]$, $\int_a^b f(x)dx$ always exists and can be computed by

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

for any choice of the x_i^* in $[x_[i - 1, x_i]$ with $\delta x = \frac{b-a}{n}$ and $x_i a + i\Delta x$. That is, P partitions $[a, b]$ into equal length subintervals (called a **regular partition**.

Example 1 Compute the Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ for the function $f(x) = \frac{1}{x}$ on $[1, 6]$ with a regular partition into $n = 5$ subintervals, and with $x_i^* = x_i$.

Solution: Note that $a = 1$, $b = 6$ and $n = 5$. Compute the following

$$\begin{aligned} \Delta x &= \frac{b-a}{n} = \frac{6-1}{5} = 1. \\ x_i &= a + i\Delta x = 1 + i, \text{ for each } i. \\ f(x_i^*) &= f(x_i) = \frac{1}{1+i}, \text{ for each } i. \end{aligned}$$

Therefore, (the answer is intentionally not simplified for students to see the algebra)

$$\sum_{i=1}^{5} f(x_i^*)\Delta x = \sum_{i=1}^{5} \frac{1}{i+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}.$$

Example 2 Compute the integral $\int_0^4 x^3 dx$ by computing Riemann sums for a regular partition.

Solution: Note that $a = 0$, $b = 4$ and $f(x) = x^3$. Use a regular partition for each positive integer n. Note that when $n \to \infty$, $|P| \to 0$. Compute the following

$$\begin{aligned} \Delta x &= \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n}. \\ x_i &= a + i\Delta x = \frac{4i}{n}, \text{ for each } i. \\ f(x_i^*) &= f(x_i) = \left(\frac{4i}{n}\right)^3 = 64\frac{i^3}{n^3}, \text{ for each } i. \end{aligned}$$

Therefore, the corresponding Riemann sum becomes (note that $\frac{1}{n^4}$ is viewed as constant with respect to the index i, and so it can be moved out of the summation sign. The last step follows from summation formulas)

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n} 64\frac{i^3}{n^3}\frac{4}{n} = \frac{256}{n^4}\sum_{i=1}^{n} i^3 = \frac{256}{n^4}\frac{n^2(n+1)^2}{4}.$$

Thus the answer is

$$\int_0^4 x^3 dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} 64\frac{i^3}{n^3}\frac{4}{n} = \lim_{n\to\infty}\frac{256}{n^4}\frac{n^2(n+1)^2}{4} = 64.$$