Compute Sums with summation notation

1. Summation Properties

(1.1) $\displaystyle\sum_{i=1}^{n} ca_n = c\sum_{i=1}^{n} a_i.$

(1.2) $\displaystyle\sum_{i=1}^{n}[a_n \pm b_n] = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i.$

2. Summation Formulas

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Example 1 Express

$$\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \frac{1}{343}$$

in summation notation:

Solution: Observe that $3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81$ and $3^5 = 343$, also $(-1)^n = 1$ if n is even and $(-1)^n = -1$ if n is odd. Thus

$$\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \frac{1}{343} = \sum_{i=1}^{5} \frac{(-1)^{i+1}}{i^2}.$$

Example 2 Express

$$\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{343}$$

in summation notation:

Solution: Observe that $2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16$ and $2^5 = 32$, and that $3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81$. Thus

$$\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{343} = \sum_{i=1}^{5} \left(\frac{2}{3}\right)^i.$$

Example 3 Express the sums $\displaystyle\sum_{i=1}^{6} \sqrt{2}i$ and $\displaystyle\sum_{k=1}^{7} \frac{(-1)^{k+1}}{k^2}$ in expanded notation.

Solution:

$$\sum_{i=1}^{6} \sqrt{2}i = \sqrt{2} + 2\sqrt{2} + 3\sqrt{2} + 4\sqrt{2} + 5\sqrt{2} + 6\sqrt{2}$$

$$\sum_{k=1}^{7} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \frac{1}{49}.$$

Example 4 Evaluate $\displaystyle\sum_{i=1}^{10}(4i-3)$.

Solution: Use Summation Properties first then apply the formulas (with $n=10$).

$$\sum_{i=1}^{10}(4i-3) = \sum_{i=1}^{10}4i - \sum_{i=1}^{10}3 = 4\sum_{i=1}^{10}i - 3\sum_{i=1}^{10}1 = 4\frac{10(10+1)}{2} - 3(10) = 220 - 30 = 190.$$

Example 5 Evaluate $\displaystyle\sum_{i=1}^{6}(i^3 - i^2)$.

Solution: Use Summation Properties first then apply the formulas (with $n=6$).

$$\sum_{i=1}^{6}(i^3 - i^2) = \sum_{i=1}^{6}i^3 - \sum_{i=1}^{6}i^2 = \frac{6^2(6+1)^2}{4} - \frac{6(6+1)(12+1)}{6} = 441 - 91 = 350.$$

Example 6 Evaluate $\displaystyle\lim_{n\to\infty}\frac{1^3 + 2^3 + \cdots + n^3}{n^4}$.

Solution: Use summation notation and summation formula, the numerator becomes

$$1^3 + 2^3 + \cdots + n^3 = \sum_{i=1}^{n}i^3 = \frac{n^2(n+1)^2}{4}.$$

Thus

$$\lim_{n\to\infty}\frac{1^3 + 2^3 + \cdots + n^3}{n^4} = \lim_{n\to\infty}\frac{n^2(n+1)^2}{4n^4} = \frac{1}{4}.$$