Compute antiderivatives with some algebraic skills and substitution techniques

Facts

(1) Many antiderivatives are not in a form that allows us to immediately apply a ready to use formula. Some algebraic skills may be needed before the antidifferentiation is done, as shown in the examples below. Please recall that $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$, $x^a x^b = x^{a+b}$, and $\frac{x^a}{x^b} = x^{a-b}$.

(2) From the chain rule we observe its antidifferentiation counter part:

$$\frac{df(u(x))}{dx} = f'(u(x))u'(x) \Longleftrightarrow \int f'(u(x))u'(x)dx = f(u(x)) + C.$$

The key of performing this art might be in the decision of choosing $u(x)$. Make sure that our choice for $u(x)$ must make $\int f'(u)du = f(u) + C$ easy to determine, and $u'(x)$ must be available in the integrand.

Example 1 Evaluate $\int \frac{2x^4 - 3x^3 + 5}{7x^2} dx$.

Solution: There is no formula that can tell us what the answer is. In order to make use of $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $(n \neq -1)$, we need some algebra to help us (the answer is intentionally not simplified for you to see the algebra):

$$\begin{aligned} \int \frac{2x^4 - 3x^3 + 5}{7x^2} dx &= \int \left(\frac{2x^4}{7x^2} - \frac{3x^3}{7x^2} + \frac{5}{7x^2} \right) dx = \frac{2}{7} \int x^2 dx - \frac{3}{7} \int x dx + \frac{5}{7} \int x^{-2} dx \\ &= \frac{2}{7}\frac{x^3}{3} - \frac{3}{7}\frac{x^2}{2} + \frac{5}{7}\frac{x^{-1}}{-1} + C. \end{aligned}$$

Example 2 Evaluate $\int \sqrt{x}(1-x)^2 dx$.

Solution: No ready to use formula can help us. To use $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $(n \neq -1)$, we need some algebra to help us (the answer is intentionally not simplified for you to see the algebra):

$$\int \sqrt{x}(1-x)^2 dx = \int x^{\frac{1}{2}}(1-2x+x^2)dx = \int x^{\frac{1}{2}} dx - 2\int x^{\frac{3}{2}} dx + \int x^{\frac{5}{2}} dx = \frac{2x^{\frac{3}{2}}}{3} - 2\frac{2x^{\frac{5}{2}}}{5} + \frac{2x^{\frac{7}{2}}}{7} + C.$$

Example 3 Evaluate $\int \frac{1}{(4x+7)^6} dx$.

Solution: Try $u = 4x + 7$. Then $u' = 4$, and $\dfrac{1}{(4x+7)^6} = u^{-6}$. Therefore,

$$\int \frac{1}{(4x+7)^6}dx = \frac{1}{4}\int \frac{1}{(4x+7)^6}4dx = \frac{1}{4}\int u^{-6}du = \frac{1}{4}\frac{u^{-5}}{-5} + C = -\frac{1}{20(4x+7)^5} + C.$$

Example 4 Evaluate $\displaystyle\int \frac{x^2}{\sqrt[3]{x^3+1}}dx$.

Solution: Try $u = x^3 + 1$. Then $u' = 3x^2$. Note that we do have an x^2 in the numerator, and $\dfrac{1}{\sqrt[3]{x^3+1}} = u^{-\frac{1}{3}}$. Therefore,

$$\int \frac{x^2}{\sqrt[3]{x^3+1}}dx = \frac{1}{3}\int \frac{1}{\sqrt[3]{x^3+1}}3x^2dx = \frac{1}{3}\int u^{-\frac{1}{3}}du = \frac{1}{3}\frac{u^{\frac{2}{3}}}{\frac{2}{3}} + C = \frac{1}{2}(x^3+1)^{\frac{2}{3}} + C.$$

Example 5 Evaluate $\displaystyle\int \cos^3 x \sin x dx$.

Solution: Try $u = \cos x$. Then $u' = -\sin x$. Note that we do have a $\sin x$ in the integrand, and $\cos^x = u^3$. Therefore,

$$\int \cos^3 x \sin x dx = -\int \cos^3 x(-\sin x)dx = -\int u^3 du = -\frac{u^4}{4} + C = -\frac{cos^4 x}{4} + C.$$

Example 6 Solve the initial value problem

$$\frac{dy}{dx} = \sqrt{x+9};\ y(-5) = \frac{1}{3}.$$

Solution: We need to find a function $y(x)$ that satisfies the differentiation equation and the initial condition. Proceed the following two steps.

(Step 1) we first compute (with $u = x + 9$),

$$y(x) = \int \sqrt{x+9}dx = \frac{2(x+9)^{\frac{3}{2}}}{3} + C.$$

(Step 2) Use the initial condition $y(-5) = \frac{1}{3}$ to determine C:

$$\frac{1}{3} = y(-5) = \frac{2(-5+9)^{\frac{3}{2}}}{3} + C = \frac{16}{3} + C \Longrightarrow C = -5.$$

Thus the answer for $y(x)$ is

$$y(x) = \frac{2(x+9)^{\frac{3}{2}}}{3} - 5.$$