Study the behavior of a function at infinity and asymptotes of the graph of a function

Facts

(1) Given a polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$ with $a_n \neq 0$, the **leading coefficient** of $f(x)$ is a_n and the **leading term** is $a_n x^n$. When $|x|$ is sufficiently large, (in other words, when $|x| \to \infty$), the behavior of $f(x)$ is similar to that of $a_n x^n$.

(2) The same conclusion can be made for power functions.

(3) Given a function $f(x)$, a vertical line $x = a$ is a **vertical asymptote** of the graph $y = f(x)$ if

$$\text{either } \lim_{x\to a^-} f(x) = \pm\infty, \text{ or } \lim_{x\to a^+} f(x) = \pm\infty.$$

(4) Given a function $f(x)$, a horizontal line $y = L$ is a **horizontal asymptote** of the graph $y = f(x)$ if

$$\text{either } \lim_{x\to -\infty} f(x) = L, \text{ or } \lim_{x\to\infty} f(x) = L.$$

(5) Given a function $f(x)$, a line $y = mx + b$ is a **slant asymptote** of the graph $y = f(x)$ if

$$\text{either } \lim_{x\to -\infty} [f(x) - (mx + b)] = 0, \text{ or } \lim_{x\to\infty} [f(x) - (mx + b)] = 0.$$

Example 1 Compute the limit

$$\lim_{x\to\infty} \frac{2x^2 + x - 2}{x^2 - 100}.$$

Solution: At infinity, a polynomial's behavior is dominated by its leading term and so

$$\lim_{x\to\infty} \frac{2x^2 + x - 2}{x^2 - 100} = \lim_{x\to\infty} \frac{2x^2}{x^2} = \lim_{x\to\infty} 2 = 2.$$

Example 2 Compute the limit

$$\lim_{x\to\infty} \frac{2x + 1}{x - x\sqrt{x}}.$$

Solution: Write $x\sqrt{x} = x^{\frac{3}{2}}$. Thus in the denominator, the leading term is $-x^{\frac{3}{2}}$.

$$\lim_{x\to\infty} \frac{2x + 1}{x - x^{\frac{3}{2}}} = \lim_{x\to\infty} \frac{2x}{-x^{\frac{3}{2}}} = \lim_{x\to\infty} \frac{2}{-x^{\frac{1}{2}}} = 0.$$

Example 3 Compute the limit

$$\lim_{x\to -\infty} \frac{5x^3 + 1}{7x^3 + 4x^2 - 2}.$$

Solution:

$$\lim_{x\to-\infty}\frac{5x^3+1}{7x^3+4x^2-2}=\lim_{x\to-\infty}\frac{5x^3}{7x^3}=\lim_{x\to-\infty}\frac{5}{7}=\frac{5}{7}.$$

Example 4 Determine the asymptotes of the graph of $f(x)=\dfrac{1}{x^2-9}$.

Solution: We observe that the domain of $f(x)$ is $(-\infty,-3)$, $(-3,3)$ and $(3,\infty)$.

To find the **vertical asymptotes**, we look at the discontinuities of $f(x)$. From the domain of $f(x)$, we see that $x=-3$ and $x=3$ are discontinuities of $f(x)$. Compute the limits

$$\lim_{x\to-3^-}\frac{1}{x^2-9}=+\infty,\ \lim_{x\to-3^+}\frac{1}{x^2-9}=-\infty,\ \lim_{x\to3^-}\frac{1}{x^2-9}=-\infty,\ \text{and}\ \lim_{x\to3^+}\frac{1}{x^2-9}=+\infty.$$

We conclude that $x=-3$ and $x=3$ are vertical asymptotes of $f(x)$.

To find the **horizontal asymptotes**, we compute the limits

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{1}{x^2-9}=0,\ \text{and}\ \lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{1}{x^2-9}=0.$$

Thus $y=0$ is the only horizontal asymptote of $f(x)$.