Applications of the second derivative: the concavity of the graph and the second derivative test

Facts Let f be a function on an interval I.
(1) Suppose that both f' and f'' exist on I. If $f''(x) > 0$ ($f''(x) < 0$, respectively) on I, then f is **concave upward** (**concave downward**, respectively) at each point in I.
(2) Let c be a point inside I and $f''(x)$ exists for the points near c. If f is concave upward on one side of c and concave downward on the other side of c, then the point $(c, f(c))$ is an **inflection point** of the graph of f. Note that an inflection point may occur at $x = c$ if $f''(c) = 0$ or if $f''(c)$ does not exist but f is continuous at $x = c$.
(3) (**The Second Derivative Test**) If for some point c inside I, $f'(c) = 0$ and $f''(x) > 0$ ($f''(x) < 0$, respectively) on I, then $f(c)$ is a local minimum (local maximum, respectively) value of $f(x)$ on I.

Example 1 Given a function $f(x) = x^3 - 3x^2$, do each of the following.
(1) Compute f', f'', and find critical points of f.
(2) Determine the intervals on which the graph of f is concave upward, and those on which the graph of f is concave downward.
(3) Determine the inflection point(s), if there are any.
(4) Classify the critical points.

Solution: The domain of f is $(-\infty, \infty)$.
(1) Compute

$$f'(x) = 3x^2 - 6x, f''(x) = 6x - 6 = 6(x - 1).$$

Set $f'(x) = 3x^2 - 6x = 0$, we have $3x(x - 2) = 0$, and so $x = 0$ and $x = 2$ are critical points.
(2) Set $f''(x) = 6(x - 1) = 0$, we have $x = 1$. This partitions the domain $(-\infty, \infty)$ into two intervals $(-\infty, 1)$ and $(1, \infty)$. Since $f''(x) = 6(x - 1)$, when x is in $(-\infty, 1)$, $f''(x) < 0$ and so $f(x)$ is concave downward in $(-\infty, 1)$; and when x is in $(1, \infty)$, $f''(x) > 0$ and so $f(x)$ is concave upward in $(1, \infty)$.
(3) As the concavity changes at $x = 1$, $(1, f(1)) = (1, -2)$ is an inflection point of the graph of f.
(4) The critical point $x = 2$ is in the interval $(1, \infty)$, on which $f(x)$ is concave upward, and so $f(2) = -4$ is a local minimum value of f. The critical point $x = 0$ in the interval $(-\infty, 1)$, on which $f(x)$ is concave downward, and so $f(0) = 0$ is a local maximum value of f.