Compute the higher derivatives

Facts For a function f, the derivative of f', denoted f'', is the **second derivative** of f; and the derivative of f'', denoted f''' or $f^{(3)}$, is the **third derivative** of f... With the notation

$$\begin{aligned} f'(x) &= D_x(f(x)) = \frac{df}{dx}, f''(x) = D_x(f'(x)) = D_x^2(f(x)) = \frac{d^2 f}{dx^2}, \\ f^{(3)}(x) &= D_x(f''(x)) = D_x^3(f(x)) = \frac{d^3 f}{dx^3}. \end{aligned}$$

we define the n**th derivative** of $f(x)$ to be

$$f^{(n)}(x) = D_x^n(f(x)) = \frac{d^n f}{dx^n}.$$

Example 1 Compute the first three derivatives of $f(x) = 2x^4 - 3x^3 + 6x - 17$.

Solution: Compute the derivatives term by term to get

$$\begin{aligned} f'(x) &= 8x^3 - 9x^2 + 6 \\ f''(x) &= 24x^2 - 18x \\ f'''(x) &= 48x - 18. \end{aligned}$$

Example 2 Compute the first three derivatives of $f(x) = 2x^5 + x^{\frac{3}{2}} - \frac{1}{2x}$.

Solution: Compute the derivatives term by term to get

$$\begin{aligned} f'(x) &= 10x^4 + \frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2x^2} \\ f''(x) &= 40x^3 + \frac{3}{4}x^{-\frac{1}{2}} - \frac{1}{x^3} \\ f'''(x) &= 120x^2 - \frac{3}{8}x^{-\frac{3}{2}} + \frac{3}{x^4}. \end{aligned}$$

Example 3 Compute the first three derivatives of $f(x) = \sin(x)\cos(x)$.

Solution: Apply product rule to get

$$\begin{aligned} f'(x) &= \cos^2(x) - \sin^2(x) \\ f''(x) &= -2\cos(x)\sin(x) - 2\cos(x)\sin(x) = -4\cos(x)\sin(x) \\ f'''(x) &= -4(\cos^2(x) - \sin^2(x)). \end{aligned}$$

Example 4 Compute the first three derivatives of $f(x) = x^2\cos(x)$.

Solution: Apply product rule to get

$$\begin{aligned} f'(x) &= 2x\cos(x) - x^2\sin(x) \\ f''(x) &= 2\cos(x) - 2x\sin(x) - 2x\sin(x) + x^2\cos(x) = 2\cos(x) - 4x\sin(x) + x^2\cos(x) \\ f'''(x) &= -2\sin(x) - 4\sin(x) - 4x\cos(x) + 2x\cos(x) - x^2\sin(x) \\ &= -6\sin(x) - 2x\cos(x) - x^2\sin(x). \end{aligned}$$

Example 5 Compute the first three derivatives of $f(x) = x\sqrt{x+1}$.

Solution: Write $f(x) = x(x+1)^{\frac{1}{2}}$. Apply product rule and chain rule in each step below.

$$\begin{aligned} f'(x) &= (x+1)^{\frac{1}{2}} + \frac{1}{2}x(x+1)^{-\frac{1}{2}} \\ f''(x) &= \frac{1}{2}(x+1)^{-\frac{1}{2}} + \frac{1}{2}(x+1)^{-\frac{1}{2}} - \frac{1}{4}x(x+1)^{-\frac{3}{2}} = (x+1)^{-\frac{1}{2}} - \frac{1}{4}x(x+1)^{-\frac{3}{2}} \\ f'''(x) &= -\frac{1}{2}(x+1)^{-\frac{3}{2}} - \frac{1}{4}(x+1)^{-\frac{3}{2}} + \frac{3}{8}x(x+1)^{-\frac{5}{2}} = -\frac{3}{4}(x+1)^{-\frac{3}{2}} + \frac{3}{8}x(x+1)^{-\frac{5}{2}}. \end{aligned}$$

Example 6 Given $\sin(y) = xy$, compute $\frac{dy}{dx}$ and $\frac{d^y}{dx^2}$.

Solution: We need to use implicit differentiation. View $y = y(x)$ and differentiate both sides of the equation $\sin(y) = xy$ with respect to x.

$$\cos(y)y' = y + xy', \text{ and so } \frac{dy}{dx} = y' = \frac{y}{\cos(y) - x}.$$

To compute $\frac{d^y}{dx^2}$ is to differentiate both sides of the equation $\cos(y)y' = y + xy'$ with respect to x. This yields

$$-\sin(y)y' + \cos(y)y'' = y' + y' + xy''.$$

Substitute $y' = \frac{y}{\cos(y)-x}$ to get

$$\frac{-y\sin(y)}{\cos(y) - x} + \cos(y)y'' = \frac{2y}{\cos(y) - x} + xy'',$$

and so $(\cos(y) - x)y'' = \frac{2y}{\cos(y)-x} + \frac{y\sin(y)}{\cos(y)-x}$. Hence

$$y'' = \frac{\frac{2y}{\cos(y)-x} + \frac{y\sin(y)}{\cos(y)-x}}{\cos(y) - x} = \frac{2y + y\sin(y)}{(\cos(y) - x)^2}.$$

Example 7 Given $x^2 + xy + y^2 = 3$, compute $\frac{dy}{dx}$ and $\frac{d^y}{dx^2}$.

Solution: We need to use implicit differentiation. View $y = y(x)$ and differentiate both sides of the equation $x^2 + xy + y^2 = 3$ with respect to x.

$$2x + y + xy' + 2yy' = 0, \text{ and so } \frac{dy}{dx} = y' = \frac{-2x - y}{x + 2y}.$$

One way to compute $\frac{d^y}{dx^2}$ is to differentiate both sides of the equation $2x + y + xy' + 2yy' = 0$ with respect to x. This yields

$$2 + y' + y' + xy'' + 2(y')^2 + 2yy'' = 0, \text{ or } 2 + 2y' + 2(y')^2 + xy'' + 2yy'' = 0.$$

Substitute $y' = \frac{-2x-y}{x+2y}$ to get

$$2 + 2\frac{-2x - y}{x + 2y} + 2\left(\frac{-2x - y}{x + 2y}\right)^2 + xy'' + 2yy'' = 0.$$

It follows that

$$(x + 2y)y'' = -2\frac{(x + 2y)^2 - (2x + y)(x + 2y) + 1}{(x + 2y)^2}.$$

Thus

$$y'' = -2\frac{(x + 2y)^2 - (2x + y)(x + 2y) + 1}{(x + 2y)^3}.$$

Another way to compute $\frac{d^y}{dx^2}$ is to differentiate y' (using quotient rule below):

$$\begin{aligned}
y'' &= D_x(y') = \frac{(-2 + y')(x + 2y) - (1 - 2y')(-2x - y)}{(x + 2y)^2} \\
&= \frac{(-2 + \frac{-2x-y}{x+2y})(x + 2y) + (1 - 2\frac{-2x-y}{x+2y})(2x + y)}{(x + 2y)^2} \\
&= -2\frac{(x + 2y)^2 - (2x + y)(x + 2y) + 1}{(x + 2y)^3}.
\end{aligned}$$

Example 8 Compute the first three derivatives of $f(x) = \frac{\sin(x)}{x}$.

Solution: Apply quotient rule in computing f', and quotient and product rules in computing f'' and f'''.

$$\begin{aligned}
f'(x) &= \frac{x\cos(x) - \sin(x)}{x^2} \\
f''(x) &= \frac{(\cos(x) - x\sin(x) - \cos(x))x^2 - 2x(x\cos(x) - \sin(x))}{x^4} = \frac{-x^2\sin(x) - 2x\cos(x) + 2\sin(x)}{x^3} \\
f'''(x) &= \frac{(-2x\sin(x) - x^2\cos(x) - 2\cos(x) + 2x\sin(x) + 2\cos(x))x^3}{x^6} \\
&\quad - \frac{3x^2(-x^2\sin(x) - 2x\cos(x) + 2\sin(x))}{x^6} \\
&= \frac{-x^3\cos(x) + 3x^2\sin(x) + 6x\cos(x) - 6\sin(x))}{x^4}.
\end{aligned}$$