The First Derivative Test and the classification of critical points (open interval min-max problems)

The First Derivative Test Suppose that $f(x)$ is continuous on an interval I and is differentiable in I except possibly a point c inside I.
(i) If $f'(x) > 0$ on the left side of c and $f'(x) < 0$ on the right side of c, then $f(c)$ is a **local maximum** value of $f(x)$ on I.
(ii) If $f'(x) < 0$ on the left side of c and $f'(x) > 0$ on the right side of c, then $f(c)$ is a **local minimum** value of $f(x)$ on I.

Example 1 Apply the first derivative test to classify the critical points of $f(x) = 2x^3 + 3x^2 - 36x + 17$.

Solution: Since $f(x)$ is a polynomial, $f(x)$ is differentiable (and so continuous) in its domain $(-\infty, \infty)$.

Compute
$$f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x-2)(x+3).$$
Thus the critical points of $f(x)$ are $x = 2$ and $x = -3$. These points divide the domain of $f(x)$ into three intervals: $(-\infty, -3), (-3, 2)$ and $(2, \infty)$.

Since $f'(-4) > 0$, $f'(0) < 0$ and $f'(3) > 0$, we conclude that $f'(x) > 0$ in both $(-\infty, -3)$ and $(2, \infty)$, and that $f'(x) < 0$ in $(-3, 2)$. Therefore, $f(x)$ is increasing in both $(-\infty, -3)$ and $(2, \infty)$, and $f(x)$ is decreasing in $(-3, 2)$.

By The First Derivative Test, $f(-3)$ is a local maximum value and $f(2)$ is a local minimum value of $f(x)$ in its domain.

Example 2 Apply the first derivative test to classify the critical points of $f(x) = x^2 + \frac{16}{x}$.

Solution: Note that $f(x)$ is differentiable (and so continuous) in its domain which consists of two intervals $(-\infty, 0)$ and $(0, \infty)$.

Compute
$$f'(x) = 2x - \frac{16}{x^2} = \frac{2x^3 - 16}{x^2} = \frac{2(x^3 - 8)}{x^2}.$$
Thus the only critical point of $f(x)$ is $x = 2$. This point divide the domain of $f(x)$ into three intervals: $(-\infty, 0), (0, 2)$ and $(2, \infty)$.

Since $f'(-1) < 0$, $f'(1) < 0$ and $f'(3) > 0$, we conclude that $f'(x) < 0$ in both $(-\infty, 0)$ and $(0, 1)$, and that $f'(x) > 0$ in $(2, \infty)$. Therefore, $f(x)$ is decreasing in both $(-\infty, -3)$ and $(0, 2)$, and $f(x)$ is increasing in $(2, \infty)$.

By The First Derivative Test, $f(2)$ is a local minimum value of $f(x)$ in its domain.

Example 3 Apply the first derivative test to classify the critical points of $f(x) = 4 + x^{\frac{2}{3}}$.

Solution: Note that $f(x)$ differentiable (and so continuous) in its domain $(infty, \infty)$.

Compute

$$f'(x) = \frac{2}{3}x^{\frac{-1}{3}}.$$

Thus the only critical point of $f(x)$ is $x = 0$ (at which $f'(x)$ does not exist). This point divide the domain of $f(x)$ into two intervals: $(-\infty, 0)$ and $(2, \infty)$.

Since $f'(-1) < 0$, and $f'(1) > 0$, we conclude that $f'(x) < 0$ in $(-\infty, 0)$, and that $f'(x) > 0$ in $(0, \infty)$. Therefore, $f(x)$ is decreasing in $(-\infty, 0)$, and $f(x)$ is increasing in $(0, \infty)$.

By The First Derivative Test, $f(0)$ is a local minimum value of $f(x)$ in its domain.

Example 4 Find the point (x, y) on the graph $y = 4 - x^2$ that is closest to the point $(3, 4)$.

Solution: By the distance formula, the distance we want to minimize of

$$d(x) = \sqrt{(x-3)^2 + (y-4)^2} = \sqrt{(x-3)^2 + (4 - x^2 - 4)^2} = \sqrt{(x-3)^2 + (x^2)^2} = \sqrt{x^4 + x^2 - 6x + 9}.$$

This amounts to minimize the function

$$f(x) = [d(x)]^2 = x^4 + x^2 - 6x + 9,$$

on its domain $(-infty, \infty)$.

Compute $f'(x) = 4x^3 + 2x - 6$. Observe that $f(1) = 0$, and so using division, we have

$$f'(x) = (x-1)(4x^2 + 4x + 6) = 2(x-1)(2x^2 + 2x + 3).$$

Note that $2x^2 + 2x - 3 > 0$ (for example, using quadratic formula to see that $2x^2 + 2x - 3 = 0$ does not have real roots) for any x. Thus $x = 1$ is the only critical point. which divides the domain of $f(x)$ in to these intervals: $(-\infty, 1)$ and $(1, \infty)$. As $f'(0) < 0$, and $f'(2) > 0$. Therefore, by the first derivative test, $f(1)$ is the only local minimum value of $f(x)$ in $(-\infty, \infty)$, and so $f(1) = 5$ is also the absolute minimum of $f(x)$ in its domain. Therefore, the point we are looking for is $(1, 3)$, and the shortest distance it $\sqrt{f(1)} = \sqrt{5}$.

Example 5 Determine two real numbers with difference 20 and minimum possible product.

Solution: Let x and y denote these two real numbers with $x \leq y$. Then $y = 20 + x$. Thus we want to minimize

$$f(x) = xy = x(20 + x) = 20x + x^2,$$

in $(-\infty, \infty)$. Note that $f'(x) = 2x + 20$, and so $x = -10$ is the only critical point. As $f'(x) < 0$ when $x < -10$ and $f'(x) > 0$ when $x > -10$. Therefore $f(-10)$ is the only local minimum value of $f(x)$ in $(-\infty, \infty)$, and so it is also the absolute minimum value of $f(x)$ in its domain. Hence $x = -10$ and $y = 10$ are the two numbers we want.