The Mean Value Theorem

Rolle's Theorem Suppose that $f(x)$ is continuous on $[a,b]$ and is differentiable in (a,b). If $f(a)=f(b)$, then there exists a point c in $(a.b)$ such that $f'(c)=0$.
The Mean Value Theorem Suppose that $f(x)$ is continuous on $[a,b]$ and is differentiable in (a,b). Then there exists a point c in $(a.b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Example 1 Show that the function $f(x)=\frac{1-x^2}{1+x^2}$ satisfies the hypotheses of Rolle's theorem on $[-1,1]$, and find all numbers c in $(-1,1)$ that satisfy the conclusion of that theorem.

Solution: Since $f(x)$ is a rational function with an always positive denominator, $f(x)$ is differentiable (and so continuous) in its domain $(-\infty,\infty)$, and in particular, $f(x)$ is continuous on $[-1,1]$, and differentiable on $(-1,1)$. As $f(-1)=0=f(1)$, we conclude that $f(x)$ satisfies the hypotheses of Rolle's theorem on $[-1,1]$.

Compute

$$f'(x)=\frac{-2x(1+x^2)-2x(1-x^2)}{(1+x^2)^2}=\frac{-4x}{(1+x^2)^2}.$$

Thus the only point c satisfying $f'(c)=0$ is $c=0$.

Example 2 Show that the function $f(x)=\sqrt{x-1}$ satisfies the hypotheses of The Mean Value Theorem on $[2,5]$, and find all numbers c in $(2,5)$ that satisfy the conclusion of that theorem.

Solution: Since $f(x)$ is a composition function of a power function ($f(u)=\sqrt{u}$) and a polynomial ($u=x-1$), $f(x)$ is continuous in its domain $[1,\infty)$, and differentiable in $(1,\infty)$; and in particular, $f(x)$ is continuous on $[2,5]$, and differentiable on $(2,5)$. Thus $f(x)$ satisfies the hypotheses of The Mean Value Theorem on $[2,5]$.

Compute $f(2)=1$ and $f(5)=2$; and $f'(x)=\frac{1}{2\sqrt{x-1}}$. As in this example, $a=2$ and $b=5$,

$$\frac{1}{2\sqrt{x-1}}=f'(x)=\frac{f(b)-f(a)}{b-a}=\frac{2-1}{5-2}=\frac{1}{3}.$$

Thus $2\sqrt{x-1}=3$, and so $4(x-1)=9$. It follows that $x=\frac{13}{4}$, and so the only point c satisfying the conclusion of that theorem is $c=\frac{13}{4}$.