Determine monotone intervals of a function

Facts: Let $f(x)$ be a function on an interval I.
(1) If for any pair of points x_1, x_2 in I with $x_1 < x_2$ we always have $f(x_1) > f(x_2)$ (respectively, $f(x_1) < f(x_2)$) then $f(x)$ is **decreasing** (respectively, **increasing**) in the interval I. If for any pair of points x_1, x_2 in I with $x_1 < x_2$ we always have $f(x_1) \le f(x_2)$ (respectively, $f(x_1) \ge f(x_2)$) then $f(x)$ is **non increasing** (respectively, **non decreasing**) in the interval I.
(2) If $f'(x) > 0$ (respectively, $f'(x) < 0$) for all x in I, then $f(x)$ is decreasing (respectively, increasing) in the interval I.
(3) To determine the monotone intervals of f (the intervals in which $f(x)$ is either always increasing or always decreasing), we can use the following process.

(Step 1) Compute $f'(x)$, and find the points at which $f'(x) = 0$ or $f'(x)$ does not exist. Let $c_1, c_2, ...$ denote these points.

(Step 2) These points $c_1, c_2, ...$ will partition the domain of $f(x)$ into intervals. Determine the sign of $f'(x)$ in each of the intervals and then apply (2) to make conclusions.

Example 1 Determine the open intervals in which the function $f(x) = 2x - \frac{1}{6}x^2 - \frac{1}{9}x^3$ is increasing and those in which $f(x)$ is decreasing.

Solution: Note that the domain of $f(x)$ is $(-\infty, \infty)$.
(Step 1) Compute $f'(x) = 2 - \frac{1}{3}x - \frac{1}{3}x^2$. Set $f'(x) = 2 - \frac{1}{3}x - \frac{1}{3}x^2 = 0$. Use 3 as a common denominator to get

$$\frac{6 - x - x^2}{3} = 0, \text{ and so } (2 - x)(3 + x) = 0.$$

Thus $c_1 = -3$ and $c_2 = 2$ are the critical points.
(Step 2) The two points -3 and 2 partitioned the domain of $f(x)$ into intervals $(-\infty, -3)$, $(-3, 2)$ and $(2, \infty)$.

Since $f'(-4) < 0$, $f'(0) > 0$ and $f'(3) < 0$, we conclude that $f'(x) < 0$ in both $(-\infty, -3)$ and $(2, \infty)$, and that $f'(x) > 0$ in $(-3, 2)$. Therefore, $f(x)$ is decreasing in both $(-\infty, -3)$ and $(2, \infty)$, and $f(x)$ is increasing in $(-3, 2)$.

Example 2 Determine the open intervals in which the function $f(x) = \frac{x}{x+1}$ is increasing and those in which $f(x)$ is decreasing.

Solution: Note that the domain of $f(x)$ is $(-\infty, -1)$ and $(-1, \infty)$.
(Step 1) Compute $f'(x) = \frac{-1}{(x+1)^2}$. Thus $f'(x) > 0$ for any x in the domain of $f(x)$.
(Step 2) Therefore, $f(x)$ is decreasing in both $(-\infty, -1)$and $(-1, \infty)$.

Example 3 Determine the open intervals in which the function $f(x) = \frac{(x-1)^2}{x^2-3}$ is increasing and those in which $f(x)$ is decreasing.

Solution: Note that the domain of $f(x)$ is $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, \sqrt{3})$ and $(\sqrt{3}, \infty)$.
(Step 1) Compute

$$\begin{aligned} f'(x) &= \frac{2(x-1)(x^2-3) - 2x(x-1)^2}{(x^2-3)^2} \\ &= \frac{(2x^3 - 2x^2 - 6x + 6) - (2x^3 - 4x^2 + 2x)}{(x^2-3)^2} \\ &= \frac{2(x^2 - 4x + 3)}{(x^2-3)^2} = \frac{2(x-1)(x-3)}{(x^2-3)^2}. \end{aligned}$$

Thus $c_1 = 1$ and $c_2 = 3$ are the critical points.
(Step 2) The two points 1 and 3 partitioned the domain of $f(x)$ into intervals $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, 1)$, $(1, \sqrt{3})$, $(\sqrt{3}, 3)$ and $(3, \infty)$.

Since $f'(-4) > 0$, $f'(0) > 0$, $f'(1.5) < 0$, $f'(2) < 0$, and $f'(4) > 0$, we conclude that $f'(x) < 0$ in the intervals $(1, \sqrt{3})$ and $(\sqrt{3}, 3)$, and that $f'(x) > 0$ in $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, 1)$, and $(3, \infty)$. Therefore, $f(x)$ is decreasing in both $(1, \sqrt{3})$ and $(\sqrt{3}, 3)$, and $f(x)$ is increasing in $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, 1)$, and $(3, \infty)$.