Compute the differential and linear approximation of a function

Facts: Let $f(x)$ be a differentiable function, and x_0 be a point in the domain of $f(x)$.
(1) The **differential** of $f(x)$ is

$$df = f'(x)dx.$$

When $y = f(x)$, we also use dy for df.
(2) A change in x, called the x-increment, is Δx, which is also denoted dx. The corresponding changes in $y = f(x)$, is

$$\Delta y = f(x + \Delta x) - f(x).$$

(3) Using the differential dy to approximate Δy is call the **linear approximation** of the function f (near the point x).
(4) Use linear approximation to estimate the value of the function $f(x)$ near $x = x_0$ with given value of Δx:

$$f(x_0 + \Delta x) \simeq f(x_0) + f'(x_0)\Delta x = f(x_0) + df(x_0).$$

The linear approximation of $f(x)$ near x_0 is often written as $L(x) = f(x_0) + f'(x_0)\Delta x$.
(5) Estimate the change of the function by the linear approximation of function $f(x)$ near $x = x_0$ with given value of Δx:

$$\text{error } = \Delta f(x_0) = f(x_0 + \Delta x) - f(x_0) \simeq f'(x_0)\Delta x = df(x_0).$$

Example 1 Compute the differential of $y = \cos\sqrt{x}$.

Solution: First compute $y' = -\sin\sqrt{x} \cdot \dfrac{1}{2\sqrt{x}}$. Thus the answer is

$$dy = -\sin\sqrt{x} \cdot \frac{1}{2\sqrt{x}}dx = -\frac{\sin\sqrt{x}}{2\sqrt{x}}dx.$$

Example 2 Given $f(x) = (1 - 2x)^{\frac{3}{2}}$. find $L(x)$ near $x = 0$.

Solution: This is the case when $x_0 = 0$ in Fact (4) above. Compute $f'(x) = (-2)\dfrac{3}{2}(1-2x)^{\frac{3}{2}} = -3(1-2x)^{\frac{3}{2}}$. Note that $f(0) = -3$. Therefore the answer is

$$L(x) = f(0) + f'(0)dx = 1 - 3dx.$$

Example 3 Use a linear approximation to estimate the number $\sqrt{80}$.

Solution: Let $f(x) = \sqrt{x}$. Then $x_0 + \Delta x = 80$ in Fact (4) above. Notice that $\sqrt{81} = 9$, we let

$x_0 = 81$ and then $\Delta x = 80 - 81 = -1$. Note that $f'(x) = \frac{1}{2\sqrt{x}}$. Thus $f'(x_0) = f'(81) = \frac{1}{2\sqrt{81}} = \frac{1}{18}$. Apply (4) to get the answer

$$\sqrt{80} \simeq \sqrt{81} + \frac{1}{18}(-1) = 9 - \frac{1}{18} = \frac{161}{18}.$$

Example 4 Use a linear approximation to estimate the number $\sin 32^o$.

Solution: Let $f(x) = \sin(x)$. Then $x_0 + \Delta x = 32^o$. Note that $\sin 30^o = \sin\frac{\pi}{6} = \frac{1}{2}$, and so we let $x_0 = 30^o = \frac{\pi}{6}$. Thus $\Delta x = 32^o - 30^o = 2^o$. When using calculus to deal with trig function values, it is recommended to convert the measure of an angle from degrees to radians. Thus

$$2^o = \frac{2^o \cdot \pi}{180^o} = \frac{\pi}{90}.$$

Note that $f'(x) = \cos(x)$ and that $f'(x_0) = f'(\frac{\pi}{6}) = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$. Apply (4) to get the answer

$$\sin 32^o \simeq \sin 30^o + \frac{\sqrt{3}}{2} \cdot \frac{\pi}{90} = \frac{1}{2} + \frac{\pi\sqrt{3}}{180} = \frac{90 + \pi\sqrt{3}}{180}.$$

Example 5 Use a linear approximation to estimate the change of the area of a square, when its edge length is decreased from 10 in to 9.8 in.

Solution: Let x denote the length of an edge of the square. Then the area is $f(x) = x^2$. Note that x changes from 10 in to 9.8 in, and so we set $x_0 = 10$ and $x_0 + \Delta x = 9.8$. It follows that $\Delta x = 9.8 - 10 = -0.2$. Note that $f'(x) = 2x$. Thus $f'(x_0) = f'(10) = 20$. Apply (5) to get the the estimated change as

$$\Delta f = f'(10)\Delta x = (20)(-0.2) = -4.$$